18.745 Introduction to Lie Algebras Fall 2004

Lecture 1 — September 7, 2004

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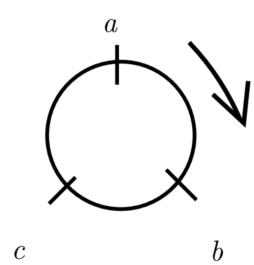
Definition 1 *(a) An* algebra *is a vector space over a field* $\mathbb{F}$*, endowed with a multiplication* ab*, which is bilinear:*

$$\begin{aligned} a(\lambda b + \mu c) &= \lambda ab + \mu ac \\ (\lambda b + \mu c)a &= \lambda ba + \mu ca \end{aligned}$$

An altebra is associative if $(ab)c = a(bc)$*.*

(b) A Lie algebra *is an algebra* $\mathfrak{g}$ *with product* $[a, b]$*, called the* bracket *of* a *and* b*, subject to two axioms:*

- *skew commutativity:* $[a, a] = 0$
- *Jacobi identity:* $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.



Remark. In a Lie algebra, one has $[b, a] = -[a, b]$.

Proof. $0 = [a + b, a + b] = [a, b] + [b, a] + \cancel{[a, a]} + \cancel{[b, b]}$ □

Examples.

1. $\mathfrak{g}$ a vector space with bracket $[a, b] = 0$. This is called an *abelian Lie algebra*.
2. $\mathbb{R}^3$ with vector multiplication $\times$ (cross product).
3. If A is an associative algebra, then $[a, b] = ab - ba$ satisfies the two identities. This Lie algebra is denoted by A_-.

   Exercise 1.1. Check the Jacobi identity on $[a, b] = ab - ba$. Moreover, this is true if A is only quasi-associative, *i.e.* $(ab)c - a(bc)$ is symmetric in a, b $(= (ba)c - b(ac))$.

Solution. First, we show that associativity implies quasi-associativity. Let $(ab)c = a(bc)$. Then $(ab)c - a(bc) = 0 = (ba)c - b(ac)$. Hence we only need to show that $[a,b]$ satisfies the Jacobi identity if A is quasi-associative.

Here are some consequences of quasi-associativity.

$$\begin{aligned}
(ab)c - a(bc) - (ba)c + b(ac) &= 0 \\
(cb)a - c(ba) - (bc)a + b(ca) &= 0 \\
(ac)b - a(cb) - (ca)b + c(ab) &= 0 \\
(ba)c - b(ac) - (ab)c + a(bc) &= 0
\end{aligned}$$

We expand the Jacobi identity, group terms, and apply quasi-associativity:

$$\begin{aligned}
& [a,[b,c]] + [b,[c,a]] + [c,[a,b]] \\
= \; & a(bc-cb) - (bc-cb)a + b(ca-ac) - (ca-ac)b + c(ab-ba) - (ab-ba)c \\
= \; & \quad [(cb)a - c(ba) - (bc)a + b(ca)] \\
& + [(ac)b - a(cb) - (ca)b + c(ab)] \\
& + [(ba)c - b(ac) - (ab)c + a(bc)] \\
= \; & 0 \text{ (since all these terms} = 0 \text{ by quasi-associativity)}
\end{aligned}$$

□

A special case is $A = \text{End}\, V$, then $A_- = \text{gl}_V$ is called the *general linear Lie algebra*. In particular, $A = \text{Mat}_n\mathbb{F}$, then $A_- = \text{gl}_n(\mathbb{F})$.

4. Any subalgebra of a Lie algebra is a Lie algebra.

   Notation: for subsets M, N of $\mathfrak{g}$ we denote $[M,N]$ the span of all commutators $[m,n]$, where $m \in M$ and $n \in N$. For example, subspace $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}'] \in \mathfrak{h}$.

Example. $\text{sl}_n(\mathbb{F}) = \{a \in \text{gl}_n(\mathbb{F}) \mid \text{tr}\ a = 0\}$

Exercise 1.2. Show that $\text{tr}[a,b] = 0$ when $a,b \in \text{gl}_n(\mathbb{F})$. Also show that if $f : \text{gl}_n(\mathbb{F}) \to \mathbb{F}$ is a linear function such that $f([a,b]) = 0, a,b \in \text{gl}_n(\mathbb{F})$, then $f(k) = c \cdot \text{tr}\ k$.

Solution. We have

$$\text{tr}\ [a,b] = \text{tr}\ ab - \text{tr}\ ba = \sum_i \sum_j a_{ji}b_{ij} - \sum_i \sum_j b_{ji}a_{ij} = 0$$

Now, any matrix $e_{ij}, i \neq j$ can be expressed as a commutator $[a,b]$ where $a,b \in \text{gl}_n(\mathbb{F})$, because $e_{ij}e_{jj} - e_{jj}e_{ij} = e_{ij}$. Hence $f(e_{ij}) = 0$ for all such matrices.

But $f(e_{ii}) = f(e_{jj}) = c$ for all i,j because $f(e_{ii}) - f(e_{jj}) = f(e_{ii}) - f(e_{jj}) = f(e_{ij}e_{ji} - e_{ji}e_{ij}) = 0$.

Any $x \in \mathrm{gl}_n(\mathbb{F})$ can be split into $x' + x''$ where x' is the sum of $e_{ij}, i \neq j$, and x'' is of the form $\sum_i \lambda_i e_{ii}$. By linearity,

$$f(x) = 0 + f(x'') = c \sum_i \lambda_i = c \cdot \mathrm{tr}.$$

□

Definition 2 *An* ideal $\mathfrak{m} \subset \mathfrak{g}$ *is a subspace such that* $[\mathfrak{m}, \mathfrak{g}] \subset \mathfrak{m}$.

Example. $\mathrm{sl}_n(\mathbb{F})$ is an ideal of $\mathrm{gl}_n(\mathbb{F})$ by Exercise 1.2.

5. Factor algebras: If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{m}$ is an ideal, then $\mathfrak{g}/\mathfrak{m}$ is a Lie algebra with bracket $[a + \mathfrak{m}, b + \mathfrak{m}] = [a, b] + \mathfrak{m}$.

6. Direct sum of two (Lie) algebras $\mathfrak{g}_1 \oplus \mathfrak{g}_2$: $[(a, b), (a_1, b_1)] = ([a, a_1], [b, b_1])$ where $a, a_1 \in \mathfrak{g}_1, b, b_1 \in \mathfrak{g}_2$.

More examples of subalgebras of gl_V: Let B be a bilinear ($\mathbb{F}$-valued) form on a vector space V over $\mathbb{F}$, define $o_{V,B} = \{a \in \mathrm{gl}_V \mid B(a(u), v) + B(u, a(v)) = 0\}$.

Exercise 1.3. Let B be a bilinear $\mathbb{F}$-valued form on a vector space V over $\mathbb{F}$. Define

$$o_{V,B} = \{a \in \mathrm{gl}_V \ \mid \ B(a(u), v) + B(u, a(v)) = 0$$

Check that this is a subalgebra of the Lie algebra gl_V.

Solution. To show that $o_{V,B}$ is a subalgebra (it is clearly a subspace), we need only show that $o_{V,B}$ is closed under the bracket. Let $x, y \in o_{V,B}$.

Consider

$$B(x(y(u)), v) = -B(y(u), x(v)) = B(u, y(x(v)))$$

But since B is bilinear and $yx = [y, x] - xy$, $B(u, y(x(v))) = B(u, [x, y] - xy) = -B(u, x(y(v))) + B(u, [x, y])$ where $B(u, [x, y])$ is 0 from above, implying that $B(x(y(u)), v) = -B(u, x(y(v)))$, giving closure:

$$B(x(y(u)), v) + B(u, x(y(v))) = 0$$

□

Important special cases: $\dim V < \infty$, B is non-degenerate (*i.e.* det of the matrix of B in some basis is non-zero).

- case 1: B is symmetric. $B(a, b) = B(b, a)$, then $o_{V,B}$ is called the orthogonal Lie algebra, notation $so_{V,B}$.

- case 2: B is skew-symmetric. $B(a, b) = -B(b, a)$, then $o_{V,B}$ is called the symplectic Lie algebra, notation $sp_{V,B}$.

Exercise 1.4. Suppose $\dim V = n$, choose a basis of V, let $so_{V,B}$ and $sp_{V,B} \subset \mathrm{gl}_n$. Let B be the matrix of the bilinear form. Show

$$\begin{aligned} so_{V,B} &= \{a \in \mathrm{gl}_n(\mathbb{F}) \mid a^T B + Ba = 0\} \\ sp_{V,B} &= \{a \in \mathrm{gl}_n(\mathbb{F}) \mid a^T B + Ba = 0\} \end{aligned}$$

Solution. Recall that gl_n is associative, and that $so_{V,B}$ is the set of $a \in A$ such that $B(u, a(v)) + B(a(u), v) = 0$ and B symmetric; similarly, $sp_{V,B}$ is the corresponding set when B is skew-symmetric. For $so_{V,B}$, we expand the definition of the bilinear form to get

$$u^T Ba(v) + (a(u))^T Bv = 0$$

and expressing a as matrix multiplication,

$$u^T (Ba)v + (au)^T Bv = u^T (Ba)v + u^T a^T Bv = 0$$

which is equivalent to the matrix condition

$$a^T B + Ba = 0$$

when B is a symmetric matrix; similarly, for a skew-symmetric B, we expand to get

$$u^T Ba(v) + (a(u))^T Bv = 0$$

and expressing a as matrix multiplication,

$$u^T (Ba)v + (au)^T Bv = u^T (Ba)v + u^T a^T Bv = 0$$

which is equivalent to the matrix condition

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□

Definition 3 *The* derived algebra *of* $\mathfrak{g}$ *is* $[\mathfrak{g}, \mathfrak{g}]$. *Obviously, this is an ideal, and hence a subalgebra.*

We now classify Lie algebras in dimensions 1 and 2.

dim 1. $\mathfrak{g} = \mathbb{F}a, [a, a] = 0$. Only the abelian one.

dim 2. $\mathfrak{g} = \mathbb{F}a + \mathbb{F}b, [\mathfrak{g}, \mathfrak{g}] = \mathbb{F}[a, b]$. case 1. $[a, b] = 0$. Then $\mathfrak{g}$ abelian. case 2. $[a, b] = c \neq 0$. So $\mathfrak{g}' = \mathbb{F}c$. Take $d \notin \mathfrak{g}'$ such that $d \neq 0$. Then $[d, c] = \alpha c$, since $\mathbb{F}c$ is an ideal, and $\alpha \neq 0$, since $\mathfrak{g}$ nonabelian. Replacing d by $\frac{1}{\alpha} d$, we get $[d, c] = c$. Thus we have a unique non-abelian Lie algebra, $[a, b] = b$.

The two most important ways to construct Lie algebras:

1. as a subalgebra (of gl_n);
2. by structure constants: choose a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$; then $[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k$. The scalars c_{ij}^k are called *structure constants.* Of course, $c_{ii}^k = 0$ and $c_{ij}^k = -c_{ij}^k$ by skew-commutativity and a quadratic equation which is the Jacobi identity.

Remark 1 *The non-abelian 2-dimensional Lie algebra is*

$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \right\} \subset gl_2(\mathbb{F})$$

If $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, *then* $[a, b] = b$.

Example 1 *The Heisenberg Lie algebra* $\mathfrak{H}_n$ *has basis* $p_i, q_i (i = 1, \ldots, n), c$. *(*$\dim \mathfrak{H}_n = 2n + 1$*) where* $[p_i, q_j] = \delta_{ij} c, [c, p_i] = 0, [c, q_i] = 0, [p_i, p_j] = 0, [q_i, q_j] = 0$. *Jacobi trivially holds. Realization by operators:* $p_i = \frac{\partial}{\partial x}, q_i = x_i, c = 1$ *on* $\mathbb{C}[x_1, \ldots, x_n]$.

The first important meaning of the Jacobi identity: Rewrite it as follows:

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]. \tag{1}$$

Definition 4 *For any algebra A, an endomorphism D is called a* derivation *if the Leibniz rule holds,* i.e.

$$D(ab) = (Da)b + a(Db).$$

Definition 5 *Given an element* $a \in \mathfrak{g}$*, define the operator ad a (adjoint) on* $\mathfrak{g}$ *by:*

$$(ad\ a)b = [a, b].$$

Equation 1 means that ad a is a derivation of the Lie algebra $\mathfrak{g}$. It is called an *inner derivation.*

Notation: Given an algebra A, denote by Der $A(\subset \text{End } A)$ the space of derivations of A.

Exercise 1.5. a) Der A is closed under the bracket in End A i.e. bracket of two derivations is a derivation, or Der A is a subalgebra of gl_A. (b) If $A = \mathfrak{g}$ is a Lie algebra, then $[D, \text{ad } a] = \text{ad } (D(a))$ for any derivation $F \in \text{Der } \mathfrak{g}$ and $a \in \mathfrak{g}$. Hence inner derivations form an ideal of the Lie algebra Der $\mathfrak{g}$.

Solution. a) Let D_1, D_2 be derivations. Consider (by parts) $[D_1, D_2]$:

$$\begin{aligned}
D_1 D_2(ab) &= D_1((D_2 a)b + a(D_2 b)) \\
&= D_1((D_2 a)b) + D_1(a(D_2 b)) \\
&= (D_1(D_2 a))b + (D_2 a)(D_1 b) + (D_1 a)(D_2 b) + a(D_1 D_2 b) \\
D_2 D_1(ab) &= (D_2(D_1 a))b + (D_1 a)(D_2 b) + (D_2 a)(D_1 b) + a(D_2 D_1 b) \\
D_1 D_2(ab) - D_2 D_1(ab) &= (D_1 D_2 a - D_2 D_1 a)b + a(D_1 D_2 b - D_2 D_1 b) \\
&= ((D_1 D_2 - D_2 D_1)a)b + a((D_1 D_2 - D_2 D_1)b)
\end{aligned}$$

showing that the bracket is a derivation.

b) Consider

$$\begin{aligned}
[D, \text{ad } a] &= (D(\text{ad } a))b - ((\text{ad } a)D)b \\
&= D[a,b] - [a, D(b)] \\
&= D(ab) - D(ba) - aD(b) + D(b)a \\
&= D(a)b + \cancel{aD(b)} - \cancel{D(b)a} - bD(a) - \cancel{aD(b)} + \cancel{D(b)a} \\
&= [D(a), b] \\
&= \text{ad } D(a)
\end{aligned}$$

which shows that inner derivations form an ideal of the Lie algebra Der $\mathfrak{g}$. $\square$