18.745 Introduction to Lie Algebras	Fall 2004
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Recall that a *trace form* on a Lie algebra $\mathfrak{g}$, given a representation π of $\mathfrak{g}$ in a finite dimensional vector space V, is defined as $(a, b)_V = \mathrm{tr}_V(\pi(a)\pi(b))$.

Definition 1. *The* Killing form *on a finite-dimensional Lie algebra* $\mathfrak{g}$ *is the trace form* $K(a, b) = tr_{\mathfrak{g}}((ad\ a)(ad\ b))$ *in the adjoint representation of* $\mathfrak{g}$.

Exercise 10.1. Show that the trace form for the defining representation of $\mathrm{gl}_n(\mathbb{F})$, $\mathrm{sl}_n(\mathbb{F})$, $\mathrm{so}_n(\mathbb{F})$, $\mathrm{sp}_n(\mathbb{F})$ is nondegenerate.

Solution. Let e_{xy} denote the nxn matrix with the element in row x, column y equal to 1 and all other elements 0. We note that $\mathrm{tr}(e_{ij}e_{kl}) = \mathrm{tr}(\delta_{jk}e_{il}) = \delta_{jk}\delta_{il}$. To show that this form is nondegenerate for a Lie algebra $\mathfrak{g} \subset \mathrm{gl}_n(\mathbb{F})$, it suffices to show for any $a \in \mathfrak{g}$ that $\mathrm{tr}(ab) \neq 0$ for some $b \in \mathfrak{g}$, so this is how we will proceed. (We will assume that $\mathrm{char}(\mathbb{F}) = 0$ for simplicity, since e.g. the trace form for $\mathrm{sl}_n(\mathbb{F})$ is degenerate when $\mathrm{char}(F) \mid n$.)

$\mathfrak{g} = \mathrm{gl}_n(\mathbb{F})$: Take nonzero $a = (a_{ij}) \in \mathfrak{g}$, and pick x, y such that $a_{xy} \neq 0$. Then $\mathrm{tr}(ae_{yx}) = a_{xy} \neq 0$.

$\mathfrak{g} = \mathrm{sl}_n(\mathbb{F})$: Choose nonzero $a = (a_{ij}) \in \mathfrak{g}$. If a is not a diagonal matrix, then we may pick $x \neq y$ such that $a_{xy} \neq 0$, and as before we have $\mathrm{tr}(ae_{yx}) \neq 0$. Otherwise, let $a = \mathrm{diag}(b_1, \ldots, b_n)$. Then $a \neq \alpha I_n$ for any α, since otherwise we would have either $a = 0$ or $\mathrm{tr}(a) = n\alpha$ with $n, \alpha \neq 0$. It follows that for some $k < n$ we must have $b_k \neq b_{k+1}$; we calculate that $\mathrm{tr}(a(e_{kk} - e_{k+1}k + 1)) = b_k - b_{k+1} \neq 0$.

$\mathfrak{g} = \mathrm{so}_n(\mathbb{F})$: Pick a basis for $\mathbb{F}^n$ such that $\mathrm{so}_n(\mathbb{F})$ is the algebra of skew-symmetric matrices, i.e. $\mathrm{so}_n(\mathbb{F}) = \{a \in \mathrm{gl}_n(\mathbb{F}) \mid a^T = -a\}$. Then $\mathfrak{g}$ has as a basis $\mathcal{B} = \{e_{ij} - e_{ji} \mid i < j\}$. We may easily calculate that for any $b_1, b_2 \in \mathcal{B}$, we have $\mathrm{tr}(b_1b_2) = -2$ if $b_1 = b_2$ and $\mathrm{tr}(b_1b_2) = 0$ otherwise. Then take nonzero $a = (a_{ij}) \in \mathfrak{g}$, and pick $x < y$ such that $a_{xy} \neq 0$; we calculate that $\mathrm{tr}(a(e_{xy} - e_{yx})) = -2a_{xy} \neq 0$.

$\mathfrak{g} = \mathrm{sp}_n(\mathbb{F})$: Note that n mus be even, and write $n = 2m$. If we pick as a representative skew-symmetric form the matrix $\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, then $\mathfrak{g}$ consists of matrices expressible in block form as $a = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$, where each block is mxm, and B and C are symmetric. Take $a \in \mathrm{sp}_n(\mathbb{F})$ of this form; if $B = C = 0$, then $\mathrm{tr}(a \cdot \begin{pmatrix} D & 0 \\ 0 & -D^T \end{pmatrix}) = 2\mathrm{tr}(AD)$, and since $A \neq 0$ and the trace form on $\mathrm{gl}_m(\mathbb{F})$ is nondegenerate, we can find $D \in \mathrm{gl}_m(\mathbb{F})$ such that $2\mathrm{tr}(AD) \neq 0$. Otherwise, assume without loss of generality that $B \neq 0$. Then $\mathrm{tr}(a \cdot \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}) = \mathrm{tr}(BE)$, so we wish to find a symmetric matrix E such that $\mathrm{tr}(BE) \neq 0$. Take the basis $\mathcal{B} = \{e_{ii}\}_{i=1}^m \cup \{e_{ij} + e_{ji} \mid i < j\}$ of the mxm symmetric matrices; then we can compute that for $b_1, b_2 \in \mathcal{B}$, $\mathrm{tr}(b_1b_2) \neq 0$ if and only if $b_1 = b_2$. Write $B = \sum_{b_i \in \mathcal{B}} c_ib_i$ for some constants $c_i \in \mathbb{F}$, and fix k such that $c_k \neq 0$; then $\mathrm{tr}(B \cdot b_k) = c_k\mathrm{tr}(b_k^2) \neq 0$, as desired.

Exercise 10.2. Show that the Killing form on $\mathrm{sl}_n(\mathbb{F})$, $\mathrm{char}(\mathbb{F}) \neq 2$, is nondegenerate if and only if $\mathrm{char}(\mathbb{F}) \not| n$.

Solution. Assume that K is nondegenerate but that $\mathrm{char}(\mathbb{F}) \mid n$. Then $\mathrm{tr}(I_n) = n = 0$ in $\mathbb{F}$, so $I_n \in \mathrm{sl}_n(\mathbb{F})$. But ad $I_n = 0$, so for all $g \in \mathrm{sl}_n(\mathbb{F})$ we have $K(I_n, g) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\ I_n)(\mathrm{ad}\ g) = \mathrm{tr}_{\mathfrak{g}} 0 = 0$, contradicting the nondegeneracy of K. Thus if K is nondegenerate, we must have $\mathrm{char}(\mathbb{F}) \not| n$.

Now suppose that $\mathrm{char}(\mathbb{F}) \not| n$. Let $\mathfrak{g} = \mathrm{sl}_n(\mathbb{F})$ have basis $\mathcal{B} = \{e_{xy} \mid x \neq y\} \cup \{e_{ii} - e_{nn} \mid i < n\}$; then for any $a, b \in g$, $K(a, b) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\ a)(\mathrm{ad}\ b)$ is the sum of the b_i-components of $(\mathrm{ad}\ a)(\mathrm{ad}\ b)b_i$ over all $b_i \in (B)$. Let $a = (a_{xy}) \in \mathfrak{g}$. We first calculate $K(a, e_{xy})$ for $x \neq y$ by computing that $[a, [e_{xy}, e_{pq}]]$ has e_{pq}-component $\delta_{py}a_{px} + \delta_{xq}a_{yq} = (\delta_{py} + \delta xq)a_{yx}$, and that $[a, [e_{xy}, e_{ii} - e_{nn}]]$ has $(e_{ii} - e_{nn})$-component $(\delta_{iy}(1 + \delta_{nx}) + \delta_{ix}(1 + \delta_{ny}))a_{yx}$, so that

$$
\begin{aligned}
K(a, e_{xy}) &= \sum_{p \neq q} \sum_{q=1}^{n} (\delta_{py} + \delta_{xq}) a_{yx} + \sum_{i=1}^{n-1} (\delta_{iy}(1 + \delta_{nx}) + \delta_{ix}(1 + \delta_{ny})) a_{yx} \\
&= 2(n-1)a_{yx} + (1 + \delta_{nx} - \delta_{ny})a_{yx} + (1 + \delta_{ny} - \delta_{nx})a_{yx} = 2na_{yx}.
\end{aligned}
$$

Hence if a is not diagonal, we may pick some $x \neq y$ such that $a_{xy} \neq 0$, and then $K(a, e_{yx}) = 2na_{xy} \neq 0$ as long as $\mathrm{char}(\mathbb{F}) \not| n$. A similar calculation yields $K(a, e_{ii} - e_{nn}) = 2n(a_{ii} - a_{nn})$. If $a \neq \alpha I_n$ for any α, then there is some $k < n$ such that $a_{kk} \neq a_{nn}$, and so $K(a, e_{kk} - e_{nn}) = 2n(a_{kk} - a_{nn}) \neq 0$. Otherwise, take α such that $a = \alpha I_n$; then $\mathrm{tr}(a) = n\alpha = 0$, so (since $\mathrm{char}(\mathbb{F}) \not| n$) we must have $\alpha = 0$ and hence $a = 0$. Since we can thus find a basis element b for any nonzero $a \in \mathfrak{g}$ such that $K(a, b) \neq 0$, it follows that K is nondegenerate. Therefore the Killing form on $\mathfrak{g} = \mathrm{sl}_n(\mathbb{F})$ is nondegenerate if and only if $\mathrm{char}(\mathbb{F}) \not| n$, as desired.

Having computed these examples, we now proceed to our first practical application of trace forms. We will need the following lemma:

Lemma 1 (Cartan's lemma). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0, and let π be a representation of $\mathfrak{g}$ in a finite-dimensional vector space V over $\mathbb{F}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, yielding generalized root and weight space decompositions $\mathfrak{g} = \oplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ and $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$. Take $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$, so that if $h = [e, f]$ then $h \in \mathfrak{g}_0 = \mathfrak{h}$. Suppose that $V_\lambda \neq 0$ for some $\lambda \in \mathfrak{h}^*$. Then $\lambda(h) = r \cdot \alpha(h)$, where r is a rational number which depends only on α and λ.*

Proof. Let $U = \oplus_{n \in \mathbb{Z}} V_{\lambda + n\alpha}$. Then U is invariant with respect to the operators $\pi(e), \pi(f)$, since $\pi(\mathfrak{g}_{\pm\alpha})V_\mu \subset V_{\pm\alpha+\mu}$. Hence we can restrict π to U and compute that $\mathrm{tr}_U(\pi(h)) = \mathrm{tr}_U(\pi([e, f])) = 0$. On the other hand, each $V_{\lambda+n\alpha}$ is invariant with respect to $h \in \mathfrak{g}_0$, so we can compute $tr_U \pi(h) = \sum_n \mathrm{tr}_{V_{\lambda+n\alpha}} \pi(h)$. But $\pi(h)|_{V_\mu}$ has an upper triangular matrix with all diagonal entries μ, so we get

$$
\begin{aligned}
0 = \mathrm{tr}_U(\pi(h)) &= \sum_n \dim(V_{\lambda+n\alpha})(\lambda + n\alpha)(h) \\
&= \lambda(h) \sum_n \dim(V_{\lambda+n\alpha}) + \alpha(h) \sum_n (n \dim(V_{\lambda+n\alpha})).
\end{aligned}
$$

Let the two sums be P and Q, respectively; they are clearly both integers, since each of the summands is, and $P = \dim(U) > 0$, so $\lambda(h) = \frac{-Q}{P}\alpha(h)$, where the ratio $\frac{-Q}{P}$ depends only on λ and α, as desired. □

We may use this to describe solvable subalgebras of gl_V in several ways in terms of trace forms.

Theorem 1 (Cartan's criterion). *Let $\mathfrak{g}$ be a subalgebra of gl_V for V a finite-dimensional vector space over a an algebraically closed field $\mathbb{F}$ of characteristic 0. Then the following are equivalent:*

1. $(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}])_V = 0$.
2. $(a, a)_V = 0$ *for any* $a \in [\mathfrak{g}, \mathfrak{g}]$.
3. $\mathfrak{g}$ *is a solvable Lie algebra.*

Proof. (1) $\Rightarrow$ (2): Take $a \in [\mathfrak{g}, \mathfrak{g}]$, and write $a = [b, c]$. Then since $(a, [b, c])_V = 0$, we have $(a, a)_V = 0$.

(2) $\Rightarrow$ (3): Suppose that $\mathfrak{g}$ is not solvable. Then the derived series of $\mathfrak{g}$ stabilizes to some nonzero subalgebra $\mathfrak{p} = \mathfrak{g}^{(N)} = \mathfrak{g}^{(N+1)} = \cdots$, so $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}$. In particular, since $\mathfrak{p} \subset \mathfrak{g}$, we have $(a, a)_V = 0$ for any $a \in [\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}$ by our assumption. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{p}$, and consider the root and weight space decompositions

$$\mathfrak{p} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{p}_\alpha, \ V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda.$$

Since $[\mathfrak{p}_\alpha, \mathfrak{p}_\beta] \subset \mathfrak{p}_{\alpha+\beta}$ and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}$, we conclude that $\mathfrak{h} = \mathfrak{p}_0 = \sum_\alpha [\mathfrak{p}_\alpha, \mathfrak{p}_{-\alpha}]$; that is, $\mathfrak{h}$ is a span of elements of the form $h_{\alpha,i} = [e_{\alpha,i}, f_{\alpha,i}]$, where $e_{\alpha,i} \in \mathfrak{p}_\alpha$ and $f_{\alpha,i} \in \mathfrak{p}_{-\alpha}$.

Suppose that $V_\lambda \neq 0$ for some fixed $\lambda \in \mathfrak{h}^*$. By Cartan's lemma, we can write $\lambda(h_{\alpha,i}) = r_{\alpha,\lambda}\alpha(h_{\alpha,i})$ for all α and i, where $r_{\alpha,\lambda} \in \mathbb{Q}$. Since we are assuming (2), $(h_{\alpha,i}, h_{\alpha,i})_V = 0$. But we compute that $(h_{\alpha,i}, h_{\alpha,i})_V = \sum_\lambda \lambda(h_{\alpha,i})^2 \dim(V_\lambda)$ by considering the restriction of $h_{\alpha,i}$ to each subspace V_λ as in the proof of Cartan's lemma; this is then equal to $\sum_\lambda r_{\alpha,\lambda}^2 \alpha(h_{\alpha,i})^2 \dim(V_\lambda)$ by the lemma, so $\alpha(h_{\alpha,i})^2 \left(\sum_\lambda r_{\alpha,\lambda}^2 \dim(V_\lambda)\right) = 0$. Hence, since all the $r_{\alpha,\lambda}$ are rational numbers, either $\alpha(h_{\alpha,i}) = 0$, or $r_{\alpha,\lambda} = 0$ whenever $V_\lambda \neq 0$, and in either case we have $\lambda(h_{\alpha,i}) = r_{\alpha,\lambda}\alpha(h_{\alpha,i}) = 0$ whenever $V_\lambda \neq 0$.

For any $\lambda \in \mathfrak{h}^*$ such that $V_\lambda \neq 0$, since the elements $\{h_{\alpha,i}\}$ span $\mathfrak{h}$, and $\lambda = 0$ on all of these elements, we conclude that $\lambda \equiv 0$ on $\mathfrak{h}$. Therefore the weight space decomposition of V is $V = V_0$. But for any $\alpha \neq 0$ we have $\mathfrak{p}_\alpha V_0 \subset V_\alpha = 0$, so $\mathfrak{p}_\alpha = 0$; hence $\mathfrak{p} = \mathfrak{p}_0 = \mathfrak{h}$. But if $\mathfrak{p} = \mathfrak{h}$ then $\mathfrak{p}$ is nilpotent, and so $[\mathfrak{p}, \mathfrak{p}]$ is a proper subset of $\mathfrak{p}$. This is a contradiction, so $\mathfrak{g}$ must in fact be solvable.

(3) $\Rightarrow$ (1): By a corollary to Lie's theorem, we may pick a basis of V such that all of the matrices in $\mathfrak{g}$ are upper triangular, hence all matrices in $[\mathfrak{g}, \mathfrak{g}]$ are strictly upper triangular. Then $\mathrm{tr}_V(ab) = 0$ for any $a \in \mathfrak{g}$ and $b \in [\mathfrak{g}, \mathfrak{g}]$, since ab is strictly upper triangular, and so $(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}])_V = 0$. ☐

This theorem leads almost immediately to a characterization of solvable Lie algebras, as follows:

Corollary 1. *A finite dimensional Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic 0 is solvable if and only if $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.*

Proof. We know that $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}/\mathrm{center}(\mathfrak{g})$ and $\mathrm{center}(\mathfrak{g})$ are both solvable, hence if and only if $\mathfrak{g}/\mathrm{center}(\mathfrak{g})$ is solvable. Consider the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{gl}_{\mathfrak{g}}$. Since $\ker(\mathrm{ad}) = \mathrm{center}(\mathfrak{g})$, this proof reduces to consideration of the subalgebra $\mathrm{ad}\,\mathfrak{g} \subset \mathrm{gl}_{\mathfrak{g}}$; we now apply the fact that conditions (1) and (3) of Cartan's criterion are equivalent, and we are done. ☐

Up until this point, we have restricted many of our results to Lie algebras over algebraically closed fields $\mathbb{F}$ of characteristic zero, since root space and weight space decompositions exist for such algebras. Requiring $\mathbb{F}$ to be algebraically closed is not always necessary, as we shall see:

Remark 1. *Let $\mathbb{F}$ be a field of characteristic zero which is not necessarily algebraically closed, and let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then the following are true:*

1. *Cartan's criterion and the corollary which followed it are both true over $\mathbb{F}$.*
2. *$\mathfrak{h}_0^a$ is a Cartan subalgebra of $\mathfrak{g}$ if $a \in \mathfrak{g}$ is regular.*
3. *$[\mathfrak{g}, \mathfrak{g}]$ is nilpotent if $\mathfrak{g}$ is finite-dimensional and solvable.*

In order to show this, we will introduce some notation: let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$, and let $\overline{\mathfrak{g}} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} \mathfrak{g}$.

Exercise 10.3. (a) Show that $\mathfrak{g}$ is solvable (resp. nilpotent, abelian) if and only if $\overline{\mathfrak{g}}$ is, and that $\overline{[\mathfrak{g}, \mathfrak{g}]} = [\overline{\mathfrak{g}}, \overline{\mathfrak{g}}]$. (b) Prove the above remark for $\mathbb{F}$ not algebraically closed.

Solution. Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ over $\mathbb{F}$, we claim that $[\overline{\mathfrak{g}}, \overline{\mathfrak{h}}] = \overline{[\mathfrak{g}, \mathfrak{h}]}$. Clearly since $\mathfrak{g} \subset \overline{\mathfrak{g}}$ and $\mathfrak{h} \subset \overline{\mathfrak{h}}$, we have $\overline{[\mathfrak{g}, \mathfrak{h}]} \subset \overline{[\overline{\mathfrak{g}}, \overline{\mathfrak{h}}]} = [\overline{\mathfrak{g}}, \overline{\mathfrak{h}}]$. Given bases $\{g_\alpha\}_{\alpha \in I}$ of $\mathfrak{g}$ and $\{h_\beta\}_{\beta \in J}$ of $\mathfrak{h}$, these same bases generate $\overline{\mathfrak{g}}$ and $\overline{\mathfrak{h}}$ when combined over $\overline{\mathbb{F}}$ rather than $\mathbb{F}$. Hence the set $\{[g_\alpha, h_\beta]\}_{\alpha \in I, \beta \in J}$ generates $[\overline{\mathfrak{g}}, \overline{\mathfrak{h}}]$ over $\overline{\mathbb{F}}$. But this set also generates $[\mathfrak{g}, \mathfrak{h}]$ with coefficients in $\mathbb{F}$, so when we extend this to $\overline{F}$ we get $[\overline{\mathfrak{g}}, \overline{\mathfrak{h}}] \subset \overline{[\mathfrak{g}, \mathfrak{h}]}$. Thus $[\overline{\mathfrak{g}}, \overline{\mathfrak{h}}] = \overline{[\mathfrak{g}, \mathfrak{h}]}$.

Part (a) now follows easily, since we note that $\overline{\mathfrak{g}}^i = \overline{\mathfrak{g}^i}$ and $\overline{\mathfrak{g}}^{(i)} = \overline{\mathfrak{g}^{(i)}}$ for all i. Then $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}^{(n)} = 0$ for some n; but this is equivalent to $\overline{\mathfrak{g}^{(n)}} = 0$, or $\overline{\mathfrak{g}}^{(n)} = 0$, and so $\mathfrak{g}$ is solvable iff $\overline{\mathfrak{g}}$ is. The same is true of nilpotency, as we see by replacing $g^{(n)}$ with g^n in this argument, and abelianness, which comes from the solvability argument with $n = 1$. In particular, letting $\mathfrak{h} = \mathfrak{g}$ above yields $\overline{[\mathfrak{g}, \mathfrak{g}]} = [\overline{\mathfrak{g}}, \overline{\mathfrak{g}}]$.

For part (b), we first look at Cartan's criterion. Since $\mathfrak{g}$ is solvable if and only if $\overline{\mathfrak{g}}$ is, we have the equivalent conditions (1) $(\overline{\mathfrak{g}}, [\overline{\mathfrak{g}}, \overline{\mathfrak{g}}]_V = 0$; (2) $(a, a)_V = 0$ for all $a \in [\overline{\mathfrak{g}}, \overline{\mathfrak{g}}]$; and (3) $\mathfrak{g}$ is solvable. But $(\overline{\mathfrak{g}}, [\overline{\mathfrak{g}}, \overline{\mathfrak{g}}])_V = (\overline{\mathfrak{g}}, \overline{[\mathfrak{g}, \mathfrak{g}]})_V = \overline{(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}])_V}$, so $(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}])_V = 0$ if and only if $(\overline{\mathfrak{g}}, [\overline{\mathfrak{g}}, \overline{\mathfrak{g}}])_V = 0$. Furthermore, $(a, a)_V = 0$ for all $a \in [\overline{\mathfrak{g}}, \overline{\mathfrak{g}}] = \overline{[\mathfrak{g}, \mathfrak{g}]}$ iff it does for all $a \in [\mathfrak{g}, \mathfrak{g}]$, since $[\mathfrak{g}, \mathfrak{g}]$ and $\overline{[\mathfrak{g}, \mathfrak{g}]}$ share the same basis over different fields and thus $(a, a)_V = 0$ for all a in one iff it does for all a in the other. Therefore Cartan's criterion holds for $\mathfrak{g}$ given that it does for $\overline{\mathfrak{g}}$; the corollary's proof over $\mathfrak{g}$ is identical to its original proof, since it only needed algebraic closure to satisfy Cartan's criterion.

Next, we consider the proof that $\mathfrak{h}_0^a$ is a Cartan subalgebra if a is regular. If $a \in \mathfrak{g}$ is regular, it is regular in $\overline{\mathfrak{g}}$, so we know that $\overline{\mathfrak{h}_0^a}$ is a Cartan subalgebra of $\overline{\mathfrak{g}}$, or $\overline{\mathfrak{h}_0^a} = N_{\overline{\mathfrak{g}}}(\overline{\mathfrak{h}_0^a})$. Since $[b, \overline{\mathfrak{h}_0^a}] \subset \overline{\mathfrak{h}_0^a}$ if and only if $[b, \mathfrak{h}_0^a] \subset \mathfrak{h}_0^a$ for any $b \in \mathfrak{g}$, we conclude that $N_{\mathfrak{g}}(\mathfrak{h}_0^a) = \mathfrak{h}_0^a$, and so $\mathfrak{h}_0^a$ is a Cartan subalgebra of $\mathfrak{g}$.

Finally, assume $\mathfrak{g}$ is finite dimensional and solvable. Then $\overline{\mathfrak{g}}$ is finite dimensional and solvable, so we know that $[\overline{\mathfrak{g}}, \overline{\mathfrak{g}}]$ is nilpotent. But $[\overline{\mathfrak{g}}, \overline{\mathfrak{g}}] = \overline{[\mathfrak{g}, \mathfrak{g}]}$, so $\overline{[\mathfrak{g}, \mathfrak{g}]}$ is nilpotent, hence $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Remark 2. *The basic properties of a trace form are that it is symmetric (i.e. $(a,b)_V = (b,a)_V$) and invariant (i.e. $([a,b],c)_V = (a,[b,c])_V$). The basic results on trace forms (like Cartan's criterion) fail, however, if we assume only that the bilinear forms involved are symmetric and invariant.*

We can construct an example of this as follows:

Exercise 10.4. Consider the 4-dimensional Lie algebra $\mathfrak{D} = \mathbb{F}p + \mathbb{F}q + \mathbb{F}c + \mathbb{F}d$, where $[p,q] = c$, c is central, $[d,p] = p$, and $[d,q] = d$. Construct a nondegenerate symmetric invariant bilinear form on $\mathfrak{D}$.

Solution. Define a bilinear form $B(x,y)$ on $\mathfrak{D}$ by the values $B(p,q) = B(q,p) = 1$; $B(c,d) = B(d,c) = 1$; $B = 0$ on all other pairs of basis elements; and all other values of B follow by symmetry and bilinearity. Then B is nondegenerate, since it has matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ with respect to the basis $\{p,q,c,d\}$ and this matrix is invertible. Invariance is also easy to check, so we are done.

The algebra $\mathfrak{D}$ is solvable, since $[\mathfrak{D},\mathfrak{D}] = \mathcal{H}_1$ is solvable, so by Cartan's criterion, any trace form on $\mathfrak{D}$ must satisfy $(\mathfrak{D},\mathcal{H}_1)_V = 0$. But $B(d,c) = 1$, so B cannot be a trace form.

We conclude the lecture by defining a new class of Lie algebras:

Definition 2. *A Lie algebra $\mathfrak{g}$ is called* semisimple *if it contains no nonzero solvable ideals. Equivalently, $\mathfrak{g}$ is semisimple if it contains no nonzero abelian ideals.*

Equivalence can be proved as follows: If $\mathfrak{g}$ has a nonzero abelian ideal, then it has a nonzero solvable ideal, since abelian ideals are solvable. Conversely, if $\mathfrak{g}$ has a nonzero solvable ideal $\mathfrak{h}$, take n such that $\mathfrak{h}^{(n)} \neq 0$ but $\mathfrak{h}^{(n+1)} = 0$; then $\mathfrak{h}^{(n)}$ is a nonzero abelian ideal.

In the next lecture, we'll prove that a finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic zero is semisimple if and only if the Killing form on $\mathfrak{g}$ is nondegenerate.