18.745: LECTURE 11

PROFESSOR: VICTOR KAČ
SCRIBE: CHRISTOPHER DAVIS

Properties of semi-simple Lie algebras

Exercise 11.1: In this exercise, sum will always mean sum as vector spaces:

(a) The sum of two ideals of a Lie algebra is an ideal.
(b) The sum of finitely many solvable ideals is a solvable ideal.
(c) The sum of a collection of solvable ideals of a finite dimensional Lie algebra is a solvable ideal.
(d) The sum of an ideal and a sub-algebra is a sub-algebra.

Proof. (a): Let $\mathfrak{g}$ denote a Lie algebra over a field $\mathbb{F}$, let $\mathfrak{a}, \mathfrak{b}$ be two ideals. An arbitrary element of $\mathfrak{a} + \mathfrak{b}$ has the form $c_1 a + c_2 b$, with $c_1, c_2 \in \mathbb{F}$, $a \in \mathfrak{a}$, $b \in \mathfrak{b}$. Let $g \in \mathfrak{g}$. Then

$$[g, c_1 a + c_2 b] = c_1[g, a] + c_2[g, b],$$

where the latter is clearly in $\mathfrak{a} + \mathfrak{b}$, since $\mathfrak{a}$ and $\mathfrak{b}$ are ideals.

(b): Using induction, it will suffice to show that the sum of two solvable ideals is solvable. Say $\mathfrak{a}^{(m)} = \mathfrak{b}^{(n)} = 0$. Then, considering bilinearity, $(\mathfrak{a} + \mathfrak{b})^{(m+n)}$ is made of elements which are sums of terms

$$[a_{m+n} \text{ or } b_{m+n}, [a_{m+n-1} \text{ or } b_{m+n-1}, [\ldots [a_2 \text{ or } b_2, a_1 \text{ or } b_1] \ldots].$$

Because $\mathfrak{a}$ and $\mathfrak{b}$ are ideals, each such term will lie in either $\mathfrak{a}^{(m)}$ or $\mathfrak{b}^{(n)}$, depending on whether there are at least m terms from $\mathfrak{a}$ or at least n terms from b. Thus, each term is zero, and we conclude that the sum is solvable. (It is an ideal by part (a).)

(c): We now assume that $\mathfrak{g}$ is finite dimensional, but also that we now have an arbitrary collection of ideals, $\mathfrak{a}_\alpha$. Consider these ideals as vector subspaces. We clearly have that $\sum \mathfrak{a}_\alpha$ is a subspace which contains each $\mathfrak{a}_\alpha$. We claim that we can find a finite subcollection $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ of the ideals $\mathfrak{a}_\alpha$ so that

$$\sum \mathfrak{a}_i = \sum \mathfrak{a}_\alpha.$$

Choose a non-empty ideal $\mathfrak{a}_1$. If $\mathfrak{a}_1 = \sum \mathfrak{a}_\alpha$, we are done, otherwise, pick another ideal $\mathfrak{a}_2$ which is not contained in $\mathfrak{a}_1$. If $\mathfrak{a}_1 + \mathfrak{a}_2 = \sum \mathfrak{a}_\alpha$, we are done, otherwise repeat the process. At each stage, the dimension $\sum \mathfrak{a}_i$ goes up by at least one, and as it is contained in the finite dimensional space $\mathfrak{g}$, the process must terminate in a finite number of steps. At this point, we can reduce to part (b).

(d): Let $a_1, a_2 \in \mathfrak{a}$ an ideal, $h_1, h_2 \in \mathfrak{h}$ a sub-algebra, and $c_i \in \mathbb{F}$ the field we are working over. Then

$$[c_1 a_1 + c_2 h_1, c_3 a_2 + c_4 h_2] = c_1[a_1, c_3 a_2 + c_4 h_2] + c_2 c_3 [h_1, a_2] + c_2 c_4 [h_1, h_2].$$

Date: October 10th, 2004.

The first two terms are in our ideal, and the third term is in our sub-algebra. Thus, we conclude that the sum of an ideal and a sub-algebra is indeed a sub-algebra. (The other properties follow immediately from bilinearity of the bracket.) □

Definition. *The radical $R(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ is the sum (as vector spaces) of all $\mathfrak{g}$'s solvable ideals.*

By 11.1(c), $R(\mathfrak{g})$ is a solvable ideal. As it contains all the other solvable ideals, it is necessarily the maximal solvable ideal of $\mathfrak{g}$. Recalling that we call a Lie Algebra with no non-zero solvable ideals *semi-simple*, we see that in the finite dimensional case, $\mathfrak{g}$ is semi-simple if and only if $R(g) = 0$.

Proposition. *For a finite dimensional Lie algebra $\mathfrak{g}$, $\overline{\mathfrak{g}} = \mathfrak{g}/R(\mathfrak{g})$ is a semi-simple Lie algebra.*

Proof. Consider any $g_1, g_2 \in \mathfrak{g}$, and let $\overline{g_1}, \overline{g_2}$ be their images in $\overline{\mathfrak{g}}$. We claim $\overline{[g_1, g_2]} = [\overline{g_1}, \overline{g_2}]$. But this is immediate from the fact that $R(\mathfrak{g})$ is an ideal, and thus taking different representatives of $\overline{g_1}$ and $\overline{g_2}$ and taking their bracket yields an element which differs from $[g_1, g_2]$ by an element of $R(\mathfrak{g})$.

Let $\overline{m}$ be a solvable ideal of $\overline{\mathfrak{g}}$. Let m denote its preimage in $\mathfrak{g}$. Then the above remarks show that $[\overline{m}, \overline{m}] = \overline{[m, m]}$, and applying this repeatedly shows that $\overline{m}$ is solvable if and only if $m^{(n)} \subseteq R(\mathfrak{g})$ for some n. Because $R(\mathfrak{g})$ is solvable, this in turn implies that m is solvable, and thus, $m \subseteq R(\mathfrak{g})$. Thus, our original $\overline{m}$ must be 0, and so $\overline{\mathfrak{g}}$ is indeed semi-simple. □

We will later prove the following, which provides a method for breaking the study of finite dimensional Lie algebras down into the study of (somewhat) simpler objects:

Theorem (Levi)**.** *If $\mathfrak{g}$ is a finite dimensional Lie algebra over a field $\mathbb{F}$ of characteristic 0, then there exists a semi-simple sub-algebra $\mathfrak{s}$ of $\mathfrak{g}$ such that*

$$\mathfrak{g} = \mathfrak{s} + R(\mathfrak{g}),$$

a sum as vector spaces.

The above is also a direct sum as vector spaces, because $\mathfrak{s} \cap R(\mathfrak{g})$ is contained in $R(g)$, hence is solvable, and is contained in the semi-simple $\mathfrak{s}$, and hence is 0. Such a situation is called a *semi-direct sum* of a sub-algebra $\mathfrak{s}$ and an ideal $R(\mathfrak{g})$.

Note that in such a case, we have a homomorphism of Lie algebras

$$\mathfrak{s} \to \text{Der } R(\mathfrak{g}),$$

defined by

$$s \to \text{ad } s|_{R(\mathfrak{g})}.$$

This situation is reminiscent of the semi-direct product from group theory, and motivates the following definition.

Definition. *Given a triple $(\mathfrak{s}, \mathfrak{r}, \varphi)$, with $\mathfrak{s}$, $\mathfrak{r}$ Lie algebras and φ a homomorphism $\mathfrak{s} \to \text{Der } \mathfrak{r}$, we define the semi-direct sum to be the Lie algebra $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ as follows:*

- *As a vector space, $\mathfrak{g} = \mathfrak{s} + \mathfrak{r}$, a direct sum of vector spaces.*
- *The commutator of two elements of $\mathfrak{s}$ (resp. of $\mathfrak{r}$) is defined as in $\mathfrak{s}$ (resp. in $\mathfrak{r}$).*
- *The commutator $[s, r]$, with $s \in \mathfrak{s}$ and $r \in \mathfrak{r}$, is defined to be $\varphi(s)(r)$.*

That the above is well-defined follows from the next exercise:

Exercise 11.2: The semi-direct sum $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ as defined above is a Lie algebra.

Proof. Implicit in the definition is that we extend all the rules to make the bracket bilinear and skew-commutative. It remains to check the Jacobi identity.

$$\begin{aligned}
&[\lambda_1 s_1 + \mu_1 r_1, [\lambda_2 s_2 + \mu_2 r_2, \lambda_3 s_3 + \mu_3 r_3]] + \\
&[\lambda_2 s_2 + \mu_2 r_2, [\lambda_3 s_3 + \mu_3 r_3, \lambda_1 s_1 + \mu_1 r_1]] + \\
&[\lambda_3 s_3 + \mu_3 r_3, [\lambda_1 s_1 + \mu_1 r_1, \lambda_2 s_2 + \mu_2 r_2]] = \\
&[\lambda_1 s_1 + \mu_1 r_1, \lambda_2\lambda_3[s_2, s_3] + \lambda_2\mu_3\varphi(s_2)(r_3) - \lambda_3\mu_2\varphi(s_3)(r_2) + \mu_2\mu_3[r_2, r_3]] + \\
&[\lambda_2 s_2 + \mu_2 r_2, -\lambda_1\lambda_3[s_1, s_3] + \mu_1\lambda_3\varphi(s_3)(r_1) - \lambda_1\mu_3\varphi(s_1)(r_3) - \mu_1\mu_3[r_1, r_3]] + \\
&[\lambda_3 s_3 + \mu_3 r_3, \lambda_1\lambda_1[s_1, s_2] + \lambda_1\mu_2\varphi(s_1)(r_2) - \mu_1\lambda_2\varphi(s_2)(r_1) + \mu_1\mu_2[r_1, r_2]] = \\
&\lambda_1\lambda_2\lambda_3[s_1, [s_2, s_3]] + \lambda_1\lambda_2\mu_3\varphi(s_1)(\varphi(s_2)(r_3) - \\
&\lambda_1\mu_2\lambda_3\varphi(s_1)(\varphi(s_3)(r_2) + \lambda_1\mu_2\mu_3\varphi(s_1)([r_2, r_3]) - \\
&\mu_1\lambda_2\lambda_3\varphi([s_2, s_3])(r_1) + \mu_1\lambda_2\mu_3[r_1, \varphi(s_2)(r_3)] - \\
&\mu_1\mu_2\lambda_3[r_1, (\varphi(s_3)(r_2)] + \mu_1\mu_2\mu_3[r_1, [r_2, r_3]] - \\
&\lambda_1\lambda_2\lambda_3[s_2, [s_1, s_3]] + \mu_1\lambda_2\lambda_3\varphi(s_2)(\varphi(s_3)(r_1) - \\
&\lambda_1\lambda_2\mu_3\varphi(s_2)(\varphi(s_1)(r_3) - \mu_1\lambda_2\mu_3\varphi(s_2)([r_1, r_3]) + \\
&\lambda_1\mu_2\lambda_3\varphi([s_1, s_3])(r_2) + \mu_1\mu_2\lambda_3[r_2, \varphi(s_3)(r_1)] - \\
&\lambda_1\mu_2\mu_3[r_2, (\varphi(s_1)(r_3)] - \mu_1\mu_2\mu_3[r_2, [r_1, r_3]] + \\
&\lambda_1\lambda_2\lambda_3[s_3, [s_1, s_2]] + \lambda_1\mu_2\lambda_3\varphi(s_3)(\varphi(s_1)(r_2) - \\
&\mu_1\lambda_2\lambda_3\varphi(s_3)(\varphi(s_2)(r_1) + \mu_1\mu_2\lambda_3\varphi(s_3)([r_1, r_2]) - \\
&\lambda_1\lambda_2\mu_3\varphi([s_1, s_2])(r_3) + \lambda_1\mu_2\mu_3[r_3, \varphi(s_1)(r_2)] - \\
&\mu_1\lambda_2\mu_3[r_3, (\varphi(s_2)(r_1)] + \mu_1\mu_2\mu_3[r_3, [r_1, r_2]] =
\end{aligned}$$

we can now simplify using the Jacobi identities for $\mathfrak{r}$ and $\mathfrak{s}$, and the identity $\varphi([s_1, s_2])(r_3) = \varphi(s_1)(\varphi(s_2)(r_3)) - \varphi(s_2)(\varphi(s_1)(r_3))$.

$$\begin{aligned}
&\lambda_1\mu_2\mu_3\varphi(s_1)([r_2, r_3]) + \mu_1\lambda_2\mu_3[r_1, \varphi(s_2)(r_3)] - \\
&\mu_1\mu_2\lambda_3[r_1, (\varphi(s_3)(r_2)] - \mu_1\lambda_2\mu_3\varphi(s_2)([r_1, r_3]) + \\
&+\mu_1\mu_2\lambda_3[r_2, \varphi(s_3)(r_1)] - \lambda_1\mu_2\mu_3[r_2, (\varphi(s_1)(r_3)] - \\
&\mu_1\mu_2\lambda_3\varphi(s_3)([r_1, r_2]) + \lambda_1\mu_2\mu_3[r_3, \varphi(s_1)(r_2)] - \mu_1\lambda_2\mu_3[r_3, (\varphi(s_2)(r_1)] = \ 0,
\end{aligned}$$

where the last equality follows from the fact that $\varphi(s_i)$ is a derivation. □

Note in particular that if $\varphi = 0$, then $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ is just the usual direct sum of $\mathfrak{s}$ and $\mathfrak{r}$.

Thus, we see that the classification of finite dimensional Lie algebras over a field of characteristic zero reduces to the determination of three things:

(1) All finite-dimensional semi-simple Lie algebras.
(2) All finite-dimensional solvable Lie algebras.
(3) For $\mathfrak{s}$ semi-simple, $\mathfrak{r}$ solvable, all homomorphisms $\phi : \mathfrak{s} \to \operatorname{Der} \mathfrak{r}$.

We will completely solve 1, but 2 is a wild problem.

Exercise 11.3: Let $\mathfrak{g} \subseteq gl_{m+n}(\mathbb{F})$ be the Lie algebra consisting of matrices of the form

$$g = \left(\begin{array}{c|c} a & b \\ \hline 0 & c \end{array}\right),$$

with $a \in gl_m$, $c \in gl_n$, and b an $m \times n$ matrix. Compute $R(\mathfrak{g})$, the complementary $\mathfrak{s}$, and describe the homomorphism $\varphi : \mathfrak{s} \to \text{Der } R(\mathfrak{g})$.

Proof. Because of the multiplication

$$\left[\left(\begin{array}{c|c} a & b \\ \hline 0 & c \end{array}\right), \left(\begin{array}{c|c} a & b \\ \hline 0 & c \end{array}\right)\right] = \left(\begin{array}{c|c} [a_0, a_1] & * \\ \hline 0 & [c_0, c_1] \end{array}\right),$$

we see that if we have an ideal $R(\mathfrak{g})$ of $\mathfrak{g}$, then the set of elements appearing in the "a" block must be an ideal of gl_m, and the elements appearing in the "c" block must be an ideal of gl_n. Furthermore, as $R(\mathfrak{g})$ is solvable, these sets must be solvable.

We will see later in lecture that the maximal solvable ideal of gl_m is simply the ideal of scalar matrices. (This is not circular, as we will not use this exercise to prove it.) We now claim that $R(\mathfrak{g})$ has no other restrictions, beyond this one. In particular, $R(\mathfrak{g})$ consists of all matrices of the form:

$$\left(\begin{array}{c|c} \lambda I_m & b \\ \hline 0 & \mu I_n \end{array}\right),$$

for all $\lambda, \mu \in \mathbb{F}$, and all $m \times n$ matrices b.

To see that this sub-algebra is solvable, consider

$$\left[\left(\begin{array}{c|c} \lambda_1 & b_1 \\ \hline 0 & \mu_1 \end{array}\right), \left(\begin{array}{c|c} \lambda_2 & b_2 \\ \hline 0 & \mu_2 \end{array}\right)\right] = \left(\begin{array}{c|c} 0 & * \\ \hline 0 & 0 \end{array}\right).$$

Thus, the matrices in the derived algebra are strictly upper-triangular, and hence our sub-algebra is solvable. Actually, the same calculation as above shows that our sub-algebra is an ideal. We simply note that not only does $[\lambda_1, \lambda_2] = 0$, but also $[\lambda_1, a_2] = 0$, for any $a_2 \in gl_m$.

For a general matrix $a \in \mathfrak{g}$, we can, through subtraction, see that

$$a \bmod R(\mathfrak{g}) \equiv \left(\begin{array}{c|c} d & 0 \\ \hline 0 & e \end{array}\right),$$

for some $d \in sl_m$ and $e \in sln$. Because we are in the characteristic zero case, we see that there is no overlap, and so the above forms the complementary algebra.

Simply by calculating the bracket, we see that

$$\varphi : \mathfrak{s} \to \text{Der } R(\mathfrak{g})$$
$$\left(\begin{array}{c|c} d & 0 \\ \hline 0 & e \end{array}\right) \to \psi_{d,e}$$

where

$$\psi_{d,e} : \left(\begin{array}{c|c} \lambda & b \\ \hline 0 & \mu \end{array}\right) \to \left(\begin{array}{c|c} 0 & db - be \\ \hline 0 & 0 \end{array}\right).$$

□

The remainder of the lecture will concern methods to identify Lie algebras as semi-simple. Recall the following definition:

Definition. *Given a subalgebra $\mathfrak{g} \subseteq gl_V$, we say that V is irreducible with respect to $\mathfrak{g}$ if the only invariant subspaces of V with respect to $\mathfrak{g}$ are 0 and V.*

Theorem. *Let V be a finite dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero. Let $\mathfrak{g} \subseteq gl_V$ be a subalgebra such that V is irreducible with respect to $\mathfrak{g}$. Then one of the following two possibilities holds: either $\mathfrak{g}$ is semi-simple, or $\mathfrak{g} = \mathfrak{s} \oplus \mathbb{F}\mathrm{I}_V$, where $\mathfrak{s}$ is semi-simple and I_V denotes the identity operator on V.*

Proof. Consider the weight space

$$V_\lambda = \{v \in V | a(v) = \lambda(a)v, a \in R(\mathfrak{g})\}.$$

(Here, the representation π is the defining representation of $\mathfrak{g} \subseteq gl_V$, and we omit it.) Because $R(\mathfrak{g})$ is solvable, Lie's Theorem assures us that the elements a have a common non-zero eigenvector. Simply because a is a linear operator on V, we have that the associated function $\lambda(a)$ is linear, and thus the weight-space is non-zero for some $\lambda \in R(\mathfrak{g})^*$.

By Lie's Lemma, V_λ is invariant with respect to $\mathfrak{g}$. From our assumptions, we can then conclude $V_\lambda = 0$ or V. But we know it's non-zero. Thus, the elements of $R(\mathfrak{g})$ act either as scalars or as zero. In the former case, the complementary semi-simple Lie algebra must then be

$$\mathfrak{s} = \{a \in \mathfrak{g} | \mathrm{tr}(a) = 0\},$$

and in this case the sum is actually a direct sum, because the scalar matrices are in the center of $\mathfrak{g}$.

□

Example. For any two non-zero vectors v_1 and v_2 in some finite dimensional vector space V, we can find an operator $g \in gl_V$ sending v_1 to v_2. Thus, any non-zero invariant subspace of gl_V must contain every non-zero vector. Clearly it must also contain the zero vector. Hence, V is irreducible with respect to gl_V.

If V is a vector space over an algebraically closed field of characteristic zero, we can then apply the above theorem. The algebra gl_V contains the scalar matrices, and in particular

$$gl_V = sl_V \oplus \mathbb{F}\mathrm{I}_V.$$

This last equation follows from the fact that sl_V is an ideal in gl_V (this follows from Exercise 1.2), $\mathbb{F}\mathrm{I}_V$, their intersection is empty, and the sum of their dimensions is the dimension of gl_V. Our theorem thus lets us conclude that sl_V is semi-simple. (It should be noted that the above argument does not work if the characteristic of our field is $p > 0$ and p divides the dimension of V, because in that case the scalar matrices are inside of sl_V.)

Exercise 11.4: Prove that $sp_{V,B}$ is always semi-simple, and $so_{V,B}$ is semi-simple if and only if the dim $V \neq 2$. (Assuming still that char $\mathbb{F} = 0$, with $\mathbb{F}$ algebraically closed.)

Proof. We first consider $sp_{V,B}$. Choose B so that B's anti-diagonal elements, i.e. the ones $(b_{n1}, \dots, b_{1n})$, are $(-1, -1, \dots, -1, 1, \dots, 1)$, with $n/2 := m$ of each. Then the matrices of $sp_{V,B}$ are those for which

$$\begin{pmatrix} -a_{n1} & \dots & -a_{m+11} & a_{m1} & \dots & a_{11} \\ -a_{n2} & \dots & -a_{m+12} & a_{m2} & \dots & a_{12} \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ -a_{nn} & \dots & -a_{m+1n} & a_{mn} & \dots & a_{1n} \end{pmatrix} +$$

$$\begin{pmatrix} a_{n1} & \dots & a_{nm+1} & \dots & a_{nn} \\ \vdots & & & & \vdots \\ a_{m+11} & \dots & a_{m+1m+1} & \dots & a_{m+1n} \\ -a_{m1} & \dots & -a_{mm+1} & \dots & -a_{mn} \\ \vdots & & & & \vdots \\ -a_{11} & & & & -a_{1n} \end{pmatrix} = 0.$$

Because the terms along the anti-diagonal are arbitrary, it's clear that if v is a vector with no zero components, then we can map v to any vector w by choosing those anti-diagonal elements suitably, and leaving the rest as zero. Consider now a vector v with $v_i = 0$, and $v_j \neq 0$. Choose a matrix in $sp_{V,B}$ with $a_{ij} = 1$. If $i + j = n + 1$, then set the rest of the entries equal to zero. Else, set the necessary entry to -1, and the rest to zero. If it appears at all, the -1 occurs in a row different from the 1, and so our matrix maps v to a vector w with $w_i = v_j \neq 0$. Repeating this process and taking some linear combination, we see that any invariant subspace containing v contains an element which has no zero entries, and as we saw above, that subspace must therefore be all of V. As the characteristic is not two, $sp_{V,B}$ does not contain the scalar matrices, and so, from the previous theorem, $sp_{V,B}$ is semi-simple.

Consider now $so_{V,B}$. Assume that $B = I_n$, and then the matrices of $so_{V,B}$ are such that $A + A^T = 0$. Say $n = 2$. Then $so_{V,B}$ is one-dimensional, and thus solvable, and in particular not semi-simple.

Consider the following equation, where the a_{ij} terms should be viewed as variables:

$$\begin{pmatrix} 0 & -a_{21} & -a_{31} & \dots & -a_{n1} \\ a_{21} & 0 & -a_{32} & & \\ a_{31} & a_{32} & \ddots & & \\ \vdots & & & & \\ a_{n1} & \dots & & & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{pmatrix}.$$

Using the same argument as above, we know that a non-zero invariant subspace contains a vector with no zero components. It's then clear that we can choose our matrix A so that $Av = (w_1, \dots, w_{i-1}, ?, w_{i+1}, \dots, w_n)^T$, i.e., we can choose all but one of the image's coordinates, and we can choose the location of the unspecified coordinate. Can we specify n linearly independent vectors in this way? Yes, as long as $n \geq 3$, because we can specify the vectors $e_1 + a_1 e_n, e_2 + a_2 e_n, \dots, e_{n-1} + a_{n-1} e_n, b_1 e_1 + e_n$. These n being linearly dependent would imply either that $b_1 = 0$, in which case we're done, or else $b_1 a_1 = 1$, and in particular, $a_1 \neq 0$. Now, choose the matrix in $so_{V,B}$ with $a_{21} = -1$, $a_{12} = 1$, and the rest of the elements zero.

This matrix maps $e_2 + b_2 e_n$ to e_1 (this is where we use that $n \geq 3$). This vector is linearly independent from the first $n-1$ we wrote above as long as $a_1 \neq 0$, which we just saw, so we have our linearly independent set of vectors, so our invariant subspace must be all of V. We conclude that V is irreducible with respect to $so_{V,B}$, and thus, if $n \geq 3$, we have that $so_{V,B}$ is semi-simple.

□

Here is another theorem useful for determining semi-simplicity:

Theorem. *A finite dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ of characteristic zero is semi-simple if and only if its Killing form is non-degenerate.*

Proof. Suppose the Killing form $K(a,b)$ is non-degenerate. Let $\mathfrak{a}$ denote an abelian ideal of $\mathfrak{g}$. To show that $\mathfrak{g}$ is semi-simple, it is equivalent to show that any such $\mathfrak{a} = 0$. Now, for $a \in \mathfrak{a}$,

$$\begin{aligned} \text{ad } a : \mathfrak{g} &\to \mathfrak{a} \\ \mathfrak{a} &\to 0, \end{aligned}$$

simply because $\mathfrak{a}$ is an ideal. Similarly, for $g \in \mathfrak{g}$, we have: ad $g : \mathfrak{a} \to \mathfrak{a}$. Write $\mathfrak{g} = \mathfrak{a} \oplus V$, a direct sum as vector spaces. We then can partially determine the form of the following matrices:

$$\begin{aligned} \text{ad } a &= \left(\begin{array}{c|c} 0 & A \\ \hline 0 & 0 \end{array}\right) \\ \text{ad } g &= \left(\begin{array}{c|c} B & C \\ \hline 0 & D \end{array}\right), \end{aligned}$$

where the first row and first column have size dim $\mathfrak{a}$ and the second row and second column have size dim $\mathfrak{V}$. Their product is a strictly upper triangular matrix, and as $g \in \mathfrak{g}$ was arbitrary, we conclude that either $a = 0$ or the Killing form is degenerate. The former must hold, lest we have a contradiction, and so our $\mathfrak{g}$ is semi-simple.

Now, for the converse, assume $\mathfrak{g}$ is semi-simple. Let

$$\mathfrak{r} = \{a \in \mathfrak{g} | K(a, \mathfrak{g}) = 0\}.$$

From lecture 9, we know that $\mathfrak{r}$ is an ideal. Considering the adjoint representation of $\mathfrak{r}$ in $gl_{\mathfrak{g}}$, then implication (2) $\Rightarrow$ (3) in Cartan's Criterion shows that ad $\mathfrak{r}$ is solvable. Then, as usual, if $(\text{ad } \mathfrak{r})^{(m)} = 0$, we know $\mathfrak{r}^{(m)}$ is in the center of $\mathfrak{g}$, and hence $\mathfrak{r}^{(m+1)} = 0$, so $\mathfrak{r}$ is solvable, which implies $\mathfrak{r} = 0$, showing that the Killing form is indeed non-degenerate.

□

The preceding theorem allows us to apply the following result to semi-simple Lie algebras.

Theorem. *If the Killing form on a finite dimensional Lie algebra is non-degenerate, then the center of $\mathfrak{g}$, $Z(\mathfrak{g})$, equals zero, and all derivations of $\mathfrak{g}$ are inner. In particular, these claims hold for a semi-simple Lie algebra over a field of characteristic zero.*

Proof. If $c \in Z(\mathfrak{g})$, then ad $c = 0$, and so

$$K(c, a) = tr_{\mathfrak{g}}(\text{ad } c \text{ ad } a) = 0,$$

hence $c = 0$. Hence the adjoint map

$$\text{ad} : \mathfrak{g} \to gl_{\mathfrak{g}}$$

is injective, and we may identify $\mathfrak{g}$ with its image. Thus, we from now on assume $\mathfrak{g} \subseteq \text{Der } \mathfrak{g} \subseteq gl_{\mathfrak{g}}$. It remains for us to prove $\mathfrak{g} = \text{Der } \mathfrak{g}$.

The Killing form on $\mathfrak{g}$ is by definition the trace form on $gl_{\mathfrak{g}}$, and its restriction to $\mathfrak{g}$ is, by our assumption, non-degenerate.

Denote by $g^{\perp}$ the set $\{D \in \text{Der } \mathfrak{g} | (D, \mathfrak{g})_{\mathfrak{g}} = 0\}$.

Exercise 11.5: If V is a finite dimensional vector space with a symmetric bilinear form B and a subspace U on which B is non-degenerate, then $V = U \oplus U^{\perp}$, where

$$U^{\perp} = \{v \in V | B(v, u) = 0 \text{ for all } u \in U\}.$$

Proof. Through a change of basis if necessary, we can assume we have a basis $\{u_1, \ldots, u_n, w_1, \ldots, w_m\}$ for V so that $\{u_1, \ldots, u_n\}$ is a basis for U on which B is the identity.

In this basis, B takes the form of the matrix

$$\left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & ? \end{array}\right),$$

where we use symmetry to fill in the top right box. From here, the assertion is obvious. □

Applying the exercise, $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$, a direct sum as vector spaces. From exercise 1.5(b), $\mathfrak{g}$ is an ideal of Der $\mathfrak{g}$. From Lecture 9, we know that the trace form is invariant, and that $\mathfrak{g}^{\perp}$ is an ideal of $\mathfrak{g}$. Thus, our direct sum as vector spaces is actually a direct sum of ideals, too, and furthermore, for any $D \in \mathfrak{g}^{\perp}$, we have $[\mathfrak{g}, D] = 0$. This means that D acts as 0 on every element $g \in \mathfrak{g}$. But since D is by definition just a derivation of $\mathfrak{g}$, we see that D is the trivial derivation, which completes the proof.

□

Finally, we end with another exercise, which shows that the previous theorem does not have a converse:

Exercise 11.6: All derivations of the two-dimensional non-abelian Lie algebra are inner, but its Killing form is degenerate.

Proof. Consider a derivation D of the non-abelian, two-dimensional Lie algebra. We know

$$Db = D[a, b] = [Da, b] + [a, Db].$$

Say $D(a) = \lambda_{11}a + \lambda_{21}b$ and $D(b) = \lambda_{12}a + \lambda_{22}b$. Then the above equation means

$$\lambda_{12}a + \lambda_{22}b = \lambda_{11}a + \lambda_{22}b,$$

which in turn implies $\lambda_{12} = \lambda_{11} = 0$.

We now simply check that

$$[\lambda_{22}a - \lambda_{21}b, a] = \lambda_{21}b$$

and

$$[\lambda_{22}a - \lambda_{21}b, b] = \lambda_{22}b,$$

which matches our derivation D. Thus we conclude that all derivations are inner. Also, note that the center of our algebra is trivial.

Now consider the Killing Form. We have

$$\text{ad } a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and ad } b = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

So, $K(b, a) = K(b, b) = 0$, and the Killing form is thus degenerate. We conclude that the previous theorem does not have a converse. $\square$