| 18.745 Introduction to Lie Algebras | Fall 2004 |
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| Lecture 12 - October 19th, 2004 | |
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**Definition.** An *Abstract Jordan Decomposition* of an element of a Lie Algebra $\mathfrak{g}$ is a decomposition of the form $a = a_s + a_n$ where $ad(a_s)$ is a semisimple operator and $ad(a_n)$ is a nilpotent operator (on $\mathfrak{g}$), and $[a_s, a_n] = 0$.

**Example.** If $\mathfrak{g} = gl_N(\mathbb{F})$, where $\mathbb{F}$ is algebraically closed, then $A = A_s + A_n \in \mathfrak{g}$ (the concrete Jordan decomposition) is also an abstract Jordan decomposition, since $ad(A_s)$ is semisimple, $ad(A_n)$ is nilpotent, and $[ad(A_s), ad(A_n)] = ad[A_s, A_n] = 0$.

**Remark.** Note that $A'_s = A_s + \lambda I, A'_n = A_n - \lambda I$ is another abstract Jordan decomposition, for any $\lambda \in \mathbb{F}$. Thus we see that the abstract Jordan decomposition is not unique itself. The uniqueness fails in this case because I is a central element.

**Claim.** An abstract Jordan decomposition is unique (if it exists) if $center(\mathfrak{g}) = 0$. For if $a = a_s + a_n = a'_s + a'_n$ are two abstract Jordan decompositions, then:

$$ad(a) = ad(a_s) + ad(a_n) = ad(a'_s) + ad(a'_n),$$

both usual Jordan decompositions of $ad(a)$. Hence by the uniqueness of the usual Jordan decomposition, $ad(a_s) = ad(a'_s)$ and $ad(a_n) = ad(a'_n)$, or $ad(a_s - a'_s) = ad(a_n - a'_n) = 0$. Then since $center(\mathfrak{g}) = 0$, we conclude that $a_s = a'_s$ and $a_n = a'_n$.

**Remark.** In some situations, an abstract Jordan decomposition may not exist, but it is difficult to construct examples of this.

**Exercise 12.1.** If $\mathfrak{g}$ is the Lie Algebra of an algebraic group over an algebraically closed field $\mathbb{F}$, then any $a \in \mathfrak{g}$ has an abstract Jordan decomposition.

*Proof.* Beyond the scope of this write up. □

**Exercise 12.2.** Using Levi's Theorem, show that any 4-dimensional Lie Algebra over an algebraically closed field of characteristic 0 is solvable or $sl_2(\mathbb{F}) \oplus \{$one dimensional abelian$\}$.

*Proof.* Let $\mathfrak{g}$ be a 4-dimensional Lie Algebra over an algebraically closed field of characteristic 0. By Levi's Theorem, there exists a semisimple subalgebra $S$ of $\mathfrak{g}$ such that $\mathfrak{g} = S \oplus R(\mathfrak{g})$, where $R(\mathfrak{g})$ is the radical of $\mathfrak{g}$, and $S \cap R(\mathfrak{g}) = 0$. If $\mathfrak{g}$ is solvable, we are done, so let us consider the case where $\mathfrak{g}$ is not solvable. In this case, $dim(S) > 0$. The cases of $dim(S) = 1$ and $dim(S) = 2$ are not possible, since then $S$ would be solvable. Suppose $dim(S) = 3$. By exercises 8.2 and 8.3, we note that the only semisimple 3-dimensional Lie Algebra (over an algebraically closed field of characteristic 0) is $sl_2(\mathbb{F})$. So $S = sl_2(\mathbb{F})$, and we have a homomorphism $sl_2(\mathbb{F}) \longrightarrow Der(R(\mathfrak{g}))$. However, $Der(R(\mathfrak{g}))$ is one dimensional, and so the kernel of this map will be an ideal of $sl_2(\mathbb{F})$ of dimension 2 or 3. If of dimension 2, the kernel would be solvable, violating the semisimplicity of $sl_2(\mathbb{F})$. So the kernel must have dimension 3, and so $sl_2(\mathbb{F})$ commutes with $R(\mathfrak{g})$. Thus $\mathfrak{g} = sl_2(\mathbb{F}) \oplus \{$one dimensional abelian$\}$.

All that remains is to handle the case where $dim(S) = 4$. We wish to show there are no semisimple 4-dimensional Lie algebras.

Assume $\mathfrak{g}$ is semisimple. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (note that $\mathfrak{h}$ must be non-zero), and write the generalized root space decomposition. Note that by Theorem 1(b) in Lecture 12, roots come in pairs, and so $dim(\mathfrak{h})$ is even.

Case $dim(\mathfrak{h}) = 2$: Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$. Choose nonzero $E \in \mathfrak{g}_\alpha$ and nonzero $F \in \mathfrak{g}_{-\alpha}$. Let $H = [E, F] \in \mathfrak{h}$. Then, the Jacobi identity,

$$[H, [E, F]] + [E, [F, H]] + [F, [H, E]] = 0,$$

implies: $[E, [F, H]] = [[H, E], F]$, with both sides in $\mathfrak{h}$. Since $ad(H)$ cannot be 0, we must have one (and thus both) of $[F, H]$ and $[H, E]$ nonzero.

Write $[H, E] = cE$, for nonzero $c \in \mathbb{F}$. This forces $[F, H] = cF$. Now we can choose a final basis vector $B$ for $\mathfrak{h}$ and subtract away a multiple of $H$ such that $[B, B] = 0$, $[B, H] = 0$, and $[B, E] = 0$. But by the Jacobi identity, we have [B,[E,F]] + [E,[F,B]] + [F,[B,E]] = 0 which implies $[F, B] = 0$, and thus $ad(B) = 0$ which contradicts semisimplicity.

Case $dim(\mathfrak{h}) = 4$: Not possible, since $\mathfrak{g}$ is not nilpotent. □

**Proposition.** *Let $\mathfrak{g}$ be a Lie Algebra (over an algebraically closed field $\mathbb{F}$) with center 0, such that all derivations of $\mathfrak{g}$ are inner. (In particular, $\mathfrak{g}$ is semisimple). Then any element of $\mathfrak{g}$ admits a (unique) abstract Jordan decomposition.*

*Proof.* Take $a \in \mathfrak{g}$. Then $A = ad(a) = A_s + A_n$ (the usual Jordan decomposition) where $A_s, A_n \in gl_{\mathfrak{g}}$, $A_s$ semisimple, $A_n$ nilpotent, and $A_s A_n = A_n A_s$. (In $End(\mathfrak{g})$). Let $\mathfrak{g} = \bigoplus \mathfrak{g}_\lambda$, $\lambda$ taken over the eigenvalues of $A_s$, be the eigenspace decomposition of $\mathfrak{g}$ with respect to $A_s$.

Let us prove that $A_s = ad(a_s)$ for some element $a_s \in \mathfrak{g}$. To do this, we must check that $A_s$ is a derivation of $\mathfrak{g}$, ie, that $A_s([x, y]) = [A_s x, y] + [x, A_s y]$. Luckily, it suffices to check this for a basis of eigenvectors of

$A_s$. Take $x \in \mathfrak{g}_\lambda$ and $y \in \mathfrak{g}_\mu$. Recall that $[x, y] \in \mathfrak{g}_{\lambda+\mu}$, since $\mathfrak{g}_\lambda$ and $\mathfrak{g}_\mu$ are generalized eigenspaces of $ad(a)$. Now it is easy to see:

$$\begin{aligned} LHS &= A_s([x, y]) = (\lambda + \mu)[x, y] \\ RHS &= [A_s x, y] + [x, A_s y] = \lambda[x, y] + \mu[x, y] = (\lambda + \mu)[x, y]. \end{aligned}$$

So $A_s$ is a derivation of $\mathfrak{g}$. Since all derivations of $\mathfrak{g}$ are inner, we have that $A_s = ad(a_s)$ for some $a_s \in \mathfrak{g}$. Hence, letting $a_n = a - a_s$, we see that $ad(a_n) = ad(a - a_s) = A - A_s = A_n$ is nilpotent, and $[ad(a_s), ad(a_n)] = 0 = ad[a_s, a_n]$. Since $center(\mathfrak{g}) = 0$, it follows that $[a_s, a_n] = 0$. Thus $a = a_s + a_n$ is an abstract Jordan decomposition of $a$. Uniqueness follows from $center(\mathfrak{g}) = 0$. □

From now on, we will assume that $\mathbb{F}$ is an algebraically closed field of characteristic 0, and that $\mathfrak{g}$ is a finite-dimensional semisimple Lie Algebra over $\mathbb{F}$.

Let $\mathfrak{h}$ be a Cartan subalgebra and let $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ be the generalized root space decomposition, where:

$$\mathfrak{g}_\alpha = \{a \in \mathfrak{g} | (ad(h) - \alpha(h)I)^N a = 0 \text{ for some } N > 0, \text{ for all } h \in \mathfrak{h}\}.$$

Recall that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, and $\mathfrak{g}_0 = \mathfrak{h}$. For semisimple algebras, we can say much more however.

**Theorem 1.**

(a) With respect to the Killing Form, $\mathfrak{g}_\alpha$, $\mathfrak{g}_\beta$ are orthogonal (ie, $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$), if $\alpha + \beta \neq 0$.

(b) $K |_{\mathfrak{g}_{-\alpha} + \mathfrak{g}_\alpha}$ is a non-degenerate bilinear form. In particular, $K |_{\mathfrak{h}}$ is non-degenerate, and $K$ defines a non-degenerate pairing of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$.

(c) $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$.

(d) $\mathfrak{h}$ consists of semisimple elements. (ie, $ad(h)$ is semisimple for all $h \in \mathfrak{h}$).

*Proof.*

(a) holds in any finite-dimensional Lie Algebra. Let $a \in \mathfrak{g}_\alpha$, $b \in \mathfrak{g}_\beta$. Then for arbitrary $\gamma$:

$$(ad(a))(ad(b))(\mathfrak{g}_\gamma) \subset \mathfrak{g}_{\gamma+\alpha+\beta}.$$

Hence, $((ad(a))(ad(b)))^N(\mathfrak{g}_\gamma) \subset \mathfrak{g}_{\gamma+N(\alpha+\beta)} = 0$ for N sufficiently large since $\alpha + \beta \neq 0$ and since there are only finitely many $\gamma$ for which $\mathfrak{g}_\gamma \neq 0$. So $(ad(a))(ad(b))$ is a nilpotent operator on $\mathfrak{g}$. Hence $tr(ad(a)ad(b)) = 0$. Thus, $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.

(b) follows from (a) and the fact that K is non-degenerate on $\mathfrak{g}$, since by (a) the kernel of $K |_{\mathfrak{g}_{-\alpha} + \mathfrak{g}_\alpha}$ lies in the kernel of $K |_{\mathfrak{g}}$.

(c) By the easy part of Cartan's Criterion, $K(\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]) = 0$, since $\mathfrak{h}$ is solvable. But by (b), $[\mathfrak{h}, \mathfrak{h}] = 0$ since $K \mid_{\mathfrak{h}}$ is non-degenerate. So $\mathfrak{h}$ is abelian. Also, $\mathfrak{h}$ is maximal abelian since it is maximal among nilpotent subalgebras (being a Cartan subalgebra).

(d) Take $h \in \mathfrak{h}$, and write the abstract Jordan decomposition $h = h_s + h_n$. For each $h' \in \mathfrak{h}$, $[ad(h), ad(h')] = ad([h, h']) = 0$, hence $ad(h_s)$ and $ad(h_n)$ commute with $ad(h')$ (see remark below). Hence since $center(\mathfrak{g}) = 0$, $[h_s, h'] = 0$ and $[h_n, h'] = 0$. Therefore, $h_s \in \mathfrak{h}$ and $h_n \in \mathfrak{h}$, since $\mathfrak{h}$ is maximal abelian. To show $h_n = 0$, we compute:

$$K(h_n, h') = tr(ad(h_n)ad(h')) = 0.$$

(since $ad(h_n)$ is nilpotent, and $ad(h')$ commutes with $ad(h_n)$, their composition is nilpotent). Hence $K(h_n, \mathfrak{h}) = 0$. Hence by (b), $h_n = 0$, and $h = h_s$ is semisimple.

□

**Remark.** We used the following fact from Linear Algebra:

If A and B are commuting operators on a finite-dimensional vector space $V$, then $A_s$ and $B$ are also commuting. Indeed, consider the generalized eigenspace decomposition for $A$, $V = \bigoplus_\lambda V_\lambda$. Each $V_\lambda$ is $B$-invariant. But $A_s$ on $V_\lambda$ is just $\lambda I$, hence $A_s B = BA_s$ on $V_\lambda$ for each $\lambda$. Hence $A_s B = BA_s$ on $V$.

Theorem 1 says that the generalized root space decompositions are just regular root spaces:

$$\mathfrak{g}_\alpha = \{a \in \mathfrak{g} | [h, a] = \alpha(h)a \text{ for all } h \in \mathfrak{h}\}$$

since all $ad(h)$ are semisimple operators.

Let $\Delta = \{\alpha \in \mathfrak{h}^* | \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$. An element of $\Delta$ is called a *root* of $\mathfrak{g}$, and $\mathfrak{g}_\alpha$ the attached root space. ($\alpha(h)$ is the eigenvalue of $ad(h)$, hence the word "root")

The root space decomposition now becomes:

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h} \oplus \Big( \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \Big), \text{ where} \\ \mathfrak{g}_\alpha &= \{a \in \mathfrak{g} | [h, a] = \alpha(h)a \text{ for all } h \in \mathfrak{h}\}, \end{aligned}$$

and $\mathfrak{h}$ is the maximal abelian subalgebra of $\mathfrak{g}$.

**Remark.** We have a linear map $\nu : \mathfrak{h} \longrightarrow \mathfrak{h}^*$ defined by $(\nu(h))(h') = K(h, h')$. But K is non-degenerate, hence $\nu$ is an isomorphism. This gives us a bilinear form on $\mathfrak{h}^*$:

$$K(\nu(h), \nu(h')) = K(h, h') = \nu(h)(h') = \nu(h')(h).$$

**Theorem 2.**

(a) If $\alpha \in \Delta$, $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$, then $[e,f] = K(e,f)\nu^{-1}(\alpha) \in \mathfrak{h}$.
(b) If $\alpha \in \Delta$, then $K(\alpha,\alpha) \neq 0$.

*Proof.*

(a) $[e,f] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{g}_0 = \mathfrak{h}$, so $[e,f] - K(e,f)\nu^{-1}(\alpha) \in \mathfrak{h}$. To prove that it is 0, we need to check that

$$K([e,f] - K(e,f)\nu^{-1}(\alpha), h') = 0, \ \forall h' \in \mathfrak{h}.$$

But this is just a computation:

$$K([e,f]-K(e,f)\nu^{-1}(\alpha),h') = K([e,f],h')-K(e,f)K(\nu^{-1}(\alpha),h') = K(e,[f,h']) - K(e,f)\alpha(h') = \alpha(h')K(e,f) - K(e,f)\alpha(h') = 0.$$

(b) Assume the contrary, that $K(\alpha,\alpha) = 0$, ie, $\alpha(\nu^{-1}(\alpha)) = 0$. Consider the following 3-dimensional subalgebra of $\mathfrak{g}$:

$$\mathbb{F}e + \mathbb{F}f + \mathbb{F}\nu^{-1}(\alpha),$$

where $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$, and $K(e,f) = 1$. (we can choose such elements by Theorem 1(b)).

By (a), $[e,f] = \nu^{-1}(\alpha)$. Also, we have:

$$\begin{aligned} [\nu^{-1}(\alpha), e] &= \alpha(\nu^{-1}(\alpha))e = 0, \\ [\nu^{-1}(\alpha), f] &= -\alpha(\nu^{-1}(\alpha))f = 0. \end{aligned}$$

Hence this 3-dimensional subalgebra is solvable (even nilpotent). Hence by Lie's Theorem, in its adjoint representation, it can be represented by upper triangular matrices in some basis. Hence its derived algebra, $\mathbb{F}\nu^{-1}(\alpha)$ can be represented by strictly upper-triangular matrices, which is impossible since $\nu^{-1}(\alpha) \in \mathfrak{h}$ is a semisimple element.

□