13 Roots of a semisimple Lie algebra

Recall that the Killing form K gives a non-degenerate bilinear pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{F}$. In addition, $[e,f] = K(e,f)\nu^{-1}(\alpha)$ for any $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$. Also, $K(\alpha,\alpha) \neq 0$ for any $\alpha \in \Delta$.

For each $\alpha \in \Delta$, choose a non-zero $E \in \mathfrak{g}_\alpha$ and an $F \in \mathfrak{g}_{-\alpha}$ such that $K(E,F) = \frac{2}{K(\alpha,\alpha)}$. Let $H = \frac{2\nu^{-1}(\alpha)}{K(\alpha,\alpha)}$. Then $[E,F] = H$, $[E,H] = -\alpha(H)E = -2E$ and $[F,H] = \alpha(H)F = 2F$ (as $\alpha(\nu^{-1}(\alpha)) = K(\alpha,\alpha)$). So the span of E, F and H is isomorphic to $sl_2(\mathbb{F})$.

Lemma 13.1 (Basic lemma for sl_2) *Let π be a representation of $sl_2(\mathbb{F})$ in a vector space V, and $v \in V$ be such that $\pi(E)v = 0$ and $\pi(H)v = \lambda v$, $\lambda \in \mathbb{F}$. Then*

1. $\pi(H)\pi(F)^n v = (\lambda - 2n)\pi(F)^n v$
2. $\pi(E)\pi(F)^n v = n(\lambda - n + 1)\pi(F)^{(n-1)} v$
3. *If V is finite-dimensional, then $\lambda \in \mathbb{Z}_+$, the vectors $\pi(F)^j v$ $(0 \le j \le \lambda)$ are linearly independent, and $\pi(F)^{(\lambda+1)}v$=0.*

Proof. (1) and (2) are proved by induction on n; we leave the details to the reader. For (3), notice that if $\lambda - n + 1$ were non-zero for all integer $n \ge 1$, then (2) would imply, by induction, that $\pi(F)^n v \neq 0$ for any $n \in \mathbb{Z}_+$. In this case (1) would mean that $\pi(H)$ has infinitely many eigenvalues, which is impossible in a finite-dimensional space. Therefore, λ is a non-negative integer.

Next, it follows from (2) by induction that $\pi(F)^j v \neq 0$ for $j \le \lambda$. Then (1) says that $\pi(F)^j v$ is an eigenvector of $\pi(H)$ that corresponds to the eigenvalue $\lambda - 2j$; therefore v, $\pi(F)v$, $\pi(F)^2 v, \ldots \pi(F)^\lambda(v)$ must be linearly independent. Finally, $\pi(F)^{\lambda+1}(j)$ must be zero, because otherwise (2) would imply, by induction, that $\pi(F)^n v$ is non-zero for infinitely many values of n, and $\pi(H)$ would have infinitely many eigenvalues, which cannot possibly happen. □

Exercise 13.1 *Prove parts (1) and (2) in the basic lemma for $sl_2(\mathbb{F})$.*

Solution. Not surprisingly, we use induction on n. Let's start with part (1). The case $n = 0$ is given. Assume this is true for some n. Then for $n+1$,

$$\pi(H)\pi(F)^{n+1}v = \pi(F)\pi(H)\pi(F)^n v - \pi([F,H])\pi(F)^n v =$$
$$= \pi(F)((\lambda - 2n)\pi(F)^n v) - 2\pi(F)\pi(F)^n v = (\lambda - 2(n+1))\pi(F)^{n+1} v$$

so, by induction, the statement is true for all n.

In part (2), as much as we'd like to, we cannot begin with $n = 0$, because $\pi(F)$ may not be invertible. So let's use $n = 1$ as the inductive base:

$$\pi(E)\pi(F)v = \pi(F)\pi(E)v - \pi([F,E])v = \pi(H)v = \lambda v$$

because $\pi(E)v = 0$, $\pi(H)v = \lambda v$ and $[E,F] = H$.

So for $n = 1$, this is true. If it is true for some n, then, for $n+1$, we write

$$\pi(E)\pi(F)^{n+1}v = \pi(F)\pi(E)\pi(F)^n(v) - \pi([F,E])\pi(F)^n(v) =$$
$$= \pi(F)(n(\lambda - n + 1)\pi(F^{n-1})v) + \pi(H)\pi(F)^n(v) =$$
$$= (n(\lambda - n + 1) + (\lambda - 2n))\pi(F)^n v = (n+1)(\lambda - (n+1) + 1)\pi(F)^n v$$

so, by induction, this statement is also true for all n.

Exercise 13.2 *Assume π is a representation of $sl_2(\mathbb{F})$ in a vector space V, and $v \in V$ is such that $\pi(F)v = 0$ and $\pi(H)v = \lambda v$. Prove that $\pi(H)\pi(E)^n v = (\lambda + 2n)\pi(E)^n v$, that $\pi(F)\pi(E)^n v = -n(\lambda + n - 1)\pi(E)^{(n-1)}v$, and that if V is given to be finite-dimensional, then $-\lambda \in \mathbb{Z}_+$, the vectors $\pi(E)^j v$, $0 \leq j \leq -\lambda$, are linearly independent, and $\pi(E)^{-\lambda+1}v = 0$.*

Solution. Rather than rewrite the proof of the basic lemma for sl_2 with minor changes (which is possible), we'll reduce this fact to the basic lemma. Namely, consider the isomorphism $\varphi : sl_2 \to sl_2$ for which

$$\varphi(E) = F, \quad \varphi(F) = E, \quad \varphi(H) = -H$$

One checks easily that φ is a Lie algebra isomorphism. Therefore, the representation π' of sl_2 in $\mathfrak{g}$ defined by $\pi' A(u) = \pi(\varphi(A))u$ is indeed a representation. Obviously, π' satisfies the assumptions of the basic lemma, except that $\pi'(H)v = -\lambda v$. Replacing all λ-s by $-\lambda$-s in the conclusions of the basic lemma, we see that, first, $\pi'(H)\pi'(F)^n v = (-\lambda - 2n)\pi'(F)^n v$. Second, $\pi'(E)\pi'(F)^n v = n(-\lambda - n + 1)\pi'(F)^{(n-1)}v$. And third, if V is finite-dimensional, then $-\lambda \in \mathbb{Z}_+$, the vectors $\pi'(F)^j v$ ($0 \leq j \leq -\lambda$) are linearly independent, and $\pi'(F)^{(-\lambda+1)}v$=0. All that's left is to replace $\pi'(E)$ by $\pi(F)$, $\pi'(F)$ by $\pi(E)$, and $\pi'(H)$ by $-\pi(H)$. Once we do that, the last three statemenbs become the three statements we need.

Theorem 13.2 *In the above notation,*

1. $\dim(\mathfrak{g}_\alpha) = 1$ *if* $\alpha \in \Delta$.
2. *If α, $\beta \in \Delta$, then $\{\beta + n\alpha : n \in \mathbb{Z}\} \cap (\Delta \cup \{0\})$ is a finite connected string. I.e. it is $\{\beta - p\alpha, \beta - (p-1)\alpha, \ldots \beta + (q-1)\alpha, \beta + q\alpha\}$, where $p, q \in \mathbb{Z}_+$ and $p - q = \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)}$. In particular, $\frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} \in \mathbb{Z}$.*
3. *If α, β, $(\alpha + \beta) \in \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$*
4. *If $\alpha \in \Delta$, then $n\alpha \in \Delta$ iff $n = \pm 1$.*

Proof. Part (1) is proved by contradiction. Assume $\dim \mathfrak{g}_\alpha > 1$. Consider the subalgebra $\mathfrak{a}_\alpha = \mathbb{F}E \oplus \mathbb{F}F \oplus \mathbb{F}H$ constructed above, where $E \in \mathfrak{g}_\alpha$, $F \in \mathfrak{g}_{-\alpha}$, $H = \frac{2\nu^{-1}(\alpha)}{K(\alpha,\alpha)} \in \mathfrak{h}$ and $[E, F] = H$, $[E, H] = -2E$, $[F, H] = 2F$. Since $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are non-degenerately paired by K, $\dim \mathfrak{g}_{-\alpha} > 1$. Therefore, there is a nonzero $v \in \mathfrak{g}_{-\alpha}$ for which $K(E, v) = 0$. For that v, $[E, v] = K(E, v)\nu^{-1}(\alpha) = 0$, and $[H, v] = -\alpha(H)v = -2v$. Observe that $\mathfrak{a}_\alpha$ is isomorphic to $sl_2(\mathbb{F})$, and it is represented in $\mathfrak{g}$ by the adjoint representation. Further, $\mathfrak{g}$ is finite-dimensional and contains a non-zero v in $\mathfrak{g}$ for which $\operatorname{ad} E(v) = 0$ and $\operatorname{ad} H(v) = -2v$. This contradicts part 3 of the basic lemma for sl_2, therefore $\dim \mathfrak{g}_\alpha = 1$.

We proceed to part (2). Let q be the largest non-negative integer for which $\beta + q\alpha \in \Delta \cup \{0\}$; it must exist, because Δ is a finite set. Pick a non-zero $v \in \mathfrak{g}_{\beta+q\alpha}$, and again consider the adjoint representation of $\mathfrak{a}_\alpha$ in $\mathfrak{g}$. Then $\operatorname{ad} E(v) \in \mathfrak{g}_{\beta+(q+1)\alpha} = \{0\}$, i.e. $\operatorname{ad} E(v) = 0$. Also,

$$\operatorname{ad} H(v) = [H, v] = ((\beta + q\alpha)(H))v = \left(\frac{2K(\beta, \alpha)}{K(\alpha, \alpha)} + 2q\right) v$$

so, once again, the basic lemma for sl_2 applies. It tells us that $\frac{2K(\beta,\alpha)}{K(\alpha,\alpha)} + 2q$ lies in $\mathbb{Z}_+$, and moreover, $\beta + q\alpha$, $\beta + (q-1)\alpha$, $\ldots$ $\beta + q\alpha - 2\left(\frac{K(\beta,\alpha)}{K(\alpha,\alpha)} + q\right)$ all lie in $\Delta \cup \{0\}$, because $\frac{2K(\beta,\alpha)}{K(\alpha,\alpha)} + 2q$, $\frac{2K(\beta,\alpha)}{K(\alpha,\alpha)} + 2q - 2$, $\ldots$ $-\frac{2K(\beta,\alpha)}{K(\alpha,\alpha)} - 2q$ are eigenvalues of $\operatorname{ad} H$.

Denote $\frac{2K(\beta,\alpha)}{K(\alpha,\alpha)} + q$ by p, and let p' be the largest non-negative integer for which $\beta - p\alpha \in \Delta \cup \{0\}$. Choose a non-zero $v' \in \mathfrak{g}_{\beta-p'\alpha}$. Then $\operatorname{ad} F(v') = 0$ (because $\operatorname{ad} F(v') \in \mathfrak{g}_{\beta-(p+1)\alpha} = \{0\}$), and $\operatorname{ad} H(v') = \left(\frac{2K(\beta,\alpha)}{K(\alpha,\alpha)} - 2p'\right) v'$. Applying the second version of the basic lemma (Exercise 13.2), we conclude that $2p' - 2\frac{K(\beta,\alpha)}{K(\alpha,\alpha)} \in \mathbb{Z}_+$, and $\beta - p'\alpha$, $\beta - (p'-1)\alpha$, $\ldots$ $\beta - p'\alpha + 2\left(-\frac{K(\beta,\alpha)}{K(\alpha,\alpha)} + p'\right)$ all lie in $\Delta \cup \{0\}$. Denote $-\frac{2K(\beta,\alpha)}{K(\alpha,\alpha)} + p'$ by q'. Since p' and q are the largest possible, we have $p' \geq p$ and $q \geq q'$. But on the other hand, $p' - q' = p - q = \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)}$. So both inequalities must be equalities, i.e. $p' = p$ and $q' = q$. Therefore, $\{\beta + n\alpha : n \in \mathbb{Z}_+\} \cap (\Delta \cup \{0\}) = \{\beta - p\alpha, \beta - (p-1)\alpha, \ldots \beta + q\alpha\}$ as claimed.

Part (3) follows from this easily. Let $\alpha, \beta \in \Delta$, and let p be the maximum non-negative integer for which $\beta - p\alpha \in \Delta$. As before, if $v \neq 0$ is an element of $\mathfrak{g}_{\alpha-p\beta}$, then $\operatorname{ad}(F)v = 0$ and $\operatorname{ad}(H)v = \left(\frac{2K(\beta,\alpha)}{K(\alpha,\alpha)} - 2p\right) v$. So by the second version of the basic lemma (Exercise 13.2), $\operatorname{ad}(E)^j v \neq 0$ if $0 \leq j \leq 2p - \frac{2K(\beta,\alpha)}{K(\alpha,\alpha)} = p + q$. Also, $q \geq 1$, because $\alpha + \beta \in \Delta$. Therefore $0 \neq \operatorname{ad}(E)^p v \in \mathfrak{g}_\beta$, $0 \neq \operatorname{ad}(E)^{p+1} v \in \mathfrak{g}_{\alpha+\beta}$, and of course, $0 \neq E \in \mathfrak{g}_\alpha$. Hence $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$, because each of these subspaces is one-dimensional.

Finally, (4) is a consequence of the above. Let $\beta = n\alpha \in \Delta$. Then, by (1), $\frac{2K(\beta,\alpha)}{K(\beta,\beta)} \in \mathbb{Z}$, i.e. $\frac{2}{n} \in \mathbb{Z}$. So all we need to show is that n can't be 2 (the same result for $-\alpha$ will then imply that n can't be -2). However, $\mathfrak{g}_{2\alpha} = [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha]$ by (3) and $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = 0$ by (1). Thus 2α is not in Δ, and n can't be 2. $\square$