18.745 Introduction to Lie Algebras	Fall 2004
Lecture 14	
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Recall that $g = h \oplus (\oplus_{\alpha\in\Delta} g_\alpha)$, dim $g_\alpha = 1$ and h is abelian. Let us compute the Killing form on h:

$K(h_1, h_2) = tr_g(ad(h_1)ad(h_2)) = tr_h(ad(h_1)ad(h_2)) + \sum tr_{g_\alpha}(ad(h_1)ad(h_2))$

Thus, $K(h_1, h_2) = \sum_\alpha \alpha(h_1)\alpha(h_2)$. On h^*, $K(\lambda, \mu) = \sum_\alpha K(\lambda, \alpha)K(\mu, \alpha)$.

Theorem 4. (a) Δ span h^* over $\mathbb{F}$.
(b) $K(\alpha, \beta) \in \mathbb{Q}$ if $\alpha, \beta \in \Delta$
(c) K on $h^*\mathbb{Q}$ is a positive-definite, $\mathbb{Q}$-valued, symmetric bilinear form.

Proof. (a) In the contrary case, there exists a nonzero $h_1 \in h$ such that $\alpha(h_1) = 0$ for all $\alpha \in \Delta$. Hence $[h_1, g_\alpha] = 0$, and $[h_1, h] = 0$ hence h_1 is in the center which is not possible.
(b) Recall that $2K(\alpha, \beta)K(\alpha, \alpha)^{-1} = p - q \in \mathbb{Z}$. But by the formula above $K(\alpha, \alpha) = \sum K(\alpha, \gamma)^2$ thus, $4K(\alpha, \alpha)^{-1} = \sum_\gamma 2K(\alpha, \gamma)K(\alpha, \alpha)^{-1} \in \mathbb{Z}$. The result follows from polarization identity.
(c) $K(\lambda, \lambda)$, $(\lambda \in h^*) = \sum_\alpha K(\alpha, \lambda)^2 \geq 0$ with equality iff all $K(\alpha, \lambda) = 0$. Hence $\lambda = 0$ since K on h is nondegenerate.

Proposition. If g is semisimple and $a \subset g$ is an ideal then K restricted to a is nondegenerate and $g = a \oplus a^\perp$. Thus, a is semisimple.

Proof. $K(a \cap a^\perp, a \cap a^\perp) = 0$, hence by Cartan's criterion $a \cap a^\perp$ is a solvable ideal. This contradicts the semisimplicity of g. Since $\dim(g) \leq \dim(a) + \dim(a^\perp)$ the result follows.

Definition. A Lie algebra is *simple* if it is not abelian and has no proper nonzero ideals.

Corollary. g is semisimple iff it is a direct sum of simple Lie algebras.

Exercise 14.1 Show that decomposition is unique up to permutation and any ideal is a subsum of the ideals in decomposition.
Solution. It suffices to prove the second statement. So let $g = \bigoplus_i g_i$ where the g_i are simple. If a is an ideal $a \cap g_i$ is either zero or g_i. We need only establish that a is homogeneous, i.e. $a = \bigoplus_i a \cap g_i$. If a has a nontrivial projection to g_i we have $[g_i, a] = g_i$ since g_i has trivial center. But $[g_i, a] \subset a$.

Let $g = g_1 \oplus g_2$ be a direct sum of semisimple Lie algebras. If h_1, h_2 are Cartan subalgebras then $h_1 \oplus h_2$ is a Cartan subalgebra. If $g_1 = h_1 \oplus \bigoplus_{\alpha\in\Delta_1} g_\alpha, g_2 = h_2 \oplus \bigoplus_{\beta\in\Delta_2} g_\beta$ then g has root space $\Delta = \Delta_1 \sqcup \Delta_2$ where we extend Δ_i by zero to the other Cartan subalgebra.

This decomposition has the property that:
(*) If $\alpha \in \Delta_1$, $\beta \in \Delta_2$ then $\beta + \alpha \notin \Delta \cup 0$.

Definition. A set of roots is *indecomposable* if there is no nontrivial decomposition such that (*) holds. Clearly an indecomposable semisimple Lie algebra must be simple. Δ is indecomposable iff for any $\alpha, \beta \in \Delta$ there exists a sequence $\alpha = \gamma_1, ..., \gamma_n = \beta$ such that $\gamma_i + \gamma_{i+1} \in \Delta$ or 0.

Example 1. $g = sl_n(n \geq 2)$, $h =$ diagonal matrices with trace zero. Let $\epsilon_i(a_{jj}) = \delta_{ij}$ and restrict ϵ_i to h. The eigenvectors for the Cartan subalgebra are E_{ij}. Note that the corresponding root is $\epsilon_i - \epsilon_j$. Hence we have root space decomposition $sl_n\mathbb{F} = h \oplus \bigoplus_{i,j} \mathbb{F}E_{ij}$. The root space $\epsilon_i - \epsilon_j$ where

$i \neq j$ is indecomposable. Indeed, $(\epsilon_i - \epsilon_j) + (\epsilon_l - \epsilon_k)$ is a root or zero if $j = l$ and $(\epsilon_i - \epsilon_j) + (\epsilon_j - \epsilon_l)$, $(\epsilon_j - \epsilon_l) + (\epsilon_l - \epsilon_k)$ are roots or zero otherwise.

Example 2. $g = so_n(\mathbb{F}) = A \in gl_n(\mathbb{F})$ such that $A^T B + BA = 0$. The best choice of B is $B_{i,n+1-i} = 1$ and zero otherwise.

Exercise 14.2 $A \in so_n(\mathbb{F})$ iff $A' = -A$ where $A'_{ij} = A_{n-j+1,n-i+1}$.
Solution. We have $(AB)_{ij} = A_{i,n+1-j}$ and $(BA)_{ij} = A_{n+1-i,j}$. From this it is easy to see that $A^T B + BA = 0$ iff $A_{n+1-j,i} + A_{n+1-i,j} = 0$. Make the substitution $i' = n + 1 - i$. We have: $A_{n+1-j,n+1-i'} + A_{i',j} = 0$ which is what we wanted.

In this case the Cartan subalgebra h is $\text{diag}(a_1, ..., a_r, -a_r, ..., -a_1)$. The roots are $F_{ij} = E_{ij} - E_{n+1-j,n+1-i}$ where the i, j are determined below. Observe that $\epsilon_i = -\epsilon_{n+1-i}$ when restricted to h. F_{ij} is a root vector with root $\epsilon_i - \epsilon_j$. Note that a basis for the root system is formed by the vectors $e_1, ..., e_r$ where either $n = 2r$ or $n = 2r + 1$. In the case $n = 2r + 1$ the roots are $\{\epsilon_i - \epsilon_j, \epsilon_i, -\epsilon_i, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j\}$ where $i \neq j$ and $i, j \leq r$. This follows from the fact that $e_{r+1} = 0$ and that $\epsilon_i = -\epsilon_{n+1-i}$.
The case $n = 2r$ is almost identical. Here the roots are $\{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j\}$ where $i \neq j$ and $i, j \leq r$.

Exercise 14.3 Do the case $r = 2n$.
Solution. See above.

Exercise 14.4 (a) $\Delta_{so_{2r+1}}$ is indecomposable iff $r \geq 1$. Thus so_{2r+1} is simple iff $r \geq 1$.
(b) $\Delta_{so_{2r}}$ is indecomposable iff $r \geq 3$.
$\Delta_{so_4} = \{\pm(\epsilon_1 - \epsilon_2)\} \cup \{\pm(\epsilon_1 + \epsilon_2)\}$, Thus, it is not simple. (so_2 is abelian).
Solution. (a) Notice that each ϵ_i is connected to ϵ_j. Note the obvious fact that roots are connected to their inverses. Furthermore, each $\pm\epsilon_i \pm \epsilon_j$ is connected to an $\pm\epsilon_i$ by choosing the sign of this last root judiciously.
(b) Note that $\pm\epsilon_i + \mu\epsilon_j$ is connected to $-\mu\epsilon_j \pm \epsilon_k$ where i, j, k are distinct and $\mu = \pm 1$ ($r \geq 3$). We have shown that one can change one index of any root without leaving the component. Thus we can connect all roots by changing one index at a time.