18.745: LECTURE 15

PROFESSOR: VICTOR KAČ
SCRIBE: MAKSIM MAYDANSKIY

Recall that for $n = 2r$ we defined $\mathrm{sp}_{2r}(\mathbb{F}) = \{A \subset \mathrm{sl}_{2r}(\mathbb{F}) | A^t J + JA^t = 0\}$, where J was any skew symmetric, but via a change of basis may be taken to be

$$J = \begin{pmatrix} 0 & 0 & 0 & \cdots & & 1 \\ 0 & 0 & & & \iddots & \\ 0 & & \ddots & 1 & & \vdots \\ \vdots & & -1 & \ddots & & 0 \\ & \iddots & & & 0 & 0 \\ -1 & & \cdots & 0 & 0 & 0 \end{pmatrix}$$

Exercise 1. *Prove that* $sp_{2r}(\mathbb{F}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | d = -a', b = b', c = c'\}$, *where* $'$ *is the transposition with respect to the antidiagonal of an* $r \times r$ *matrix.*

Note. For $\mathrm{so}_{2r}(\mathbb{F})$ we have $b' = -b$, $c' = -c$.

Solution. Denote the antidiagonal $r \times r$ matrix by j. Then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{sp}_{2r}(\mathbb{F}) \Leftrightarrow$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} + \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \Leftrightarrow$$

$$\begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} + \begin{pmatrix} jc & jd \\ -ja & -jb \end{pmatrix} = 0 \Leftrightarrow$$

$$\begin{pmatrix} -c^t j + jc & a^t j + jd \\ -d^t j - ja & b^t j - jb \end{pmatrix} = 0 \Leftrightarrow \begin{matrix} -c^t j + jc = 0 \\ a^t j + jd = 0 \\ -d^t j - ja = 0 \\ b^t j - jb = 0 \end{matrix} \Leftrightarrow \begin{matrix} c & = & -j^{-1} c^t j \\ d & = & j^{-1} a^t j \\ a & = & -j^{-1} d^t j \\ b & = & j^{-1} b^t j \end{matrix}$$

Now for $A = \{a_{i,j}\}_{i,j}$ we have $jA = \{a_{n+1-i,j}\}_{i,j}$ and $Aj = \{a_{n+1-i,n+1-j}\}_{i,j}$. Since $j^{-1} = j$, we get $j^{-1} A^t j = \{a_{n+1-i,n+1-j}\}_{i,j} = A'$, and so indeed $M \in \mathrm{sp}_{2r}(\mathbb{F}) \Leftrightarrow c = c', b = b', a = -d'$.

□

Inside $\mathrm{sp}_{2r}(\mathbb{F})$ we have

$$h = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_r & & & \\ & & & -a_r & & \\ & & & & \ddots & \\ & & & & & -a_1 \end{pmatrix}.$$

as a Cartan subalgebra. Indeed, in this basis $\mathrm{sp}_{2r}(\mathbb{F})$ contains a matrix with distinct eigenvalues, and so by a remark in one of the previous lectures the intersectio of $\mathrm{sp}_{2r}(\mathbb{F})$ and the diagonal subalgebra is a Cartan subalgebra. In light of Exercise 1, this intersection is as above.

The vectors $F_{i,j} = E_{i,j} - E_{n+1-i,n+1-j}$ for $1 \leq i, j \leq r$, $i \neq j$
and $F_{i,j} = E_{i,j} - E_{n+1-j,n+1-i}$ for $r+1 \leq j \leq n$, $1 \leq i \leq r$ are root vectors. The corresponding roots are $\Delta_{\text{sp}} = \{\varepsilon_i - \varepsilon_j$ for $i, j = 1, \ldots, r,$ with $i \neq j; \varepsilon_i + \varepsilon_j$ for $i, j = 1, \ldots, r; -\varepsilon_i - \varepsilon_j$ for $i, j = 1, \ldots, r\}$. Note that $\pm 2\varepsilon_i$ are roots.

Note. This set is indecomposable for all $r \geq 1$, hence $\text{sp}_{2r}(\mathbb{F})$ is simple.

Proof. For $r \geq 2$ we have the chains $\varepsilon_a + \varepsilon_b, -\varepsilon_b + \varepsilon_c, \varepsilon_c + \varepsilon_b$ if $c \neq b$ and $\varepsilon_a + \varepsilon_b, -\varepsilon_b + \varepsilon_a, \varepsilon_c + \varepsilon_b$ if $a \neq b$. This shows that all $\varepsilon_i + \varepsilon_j$'s are connected (if $r = 1$ this statement is vacuously true).

The chain $\varepsilon_a + \varepsilon_b, \varepsilon_a - \varepsilon_b$ for $a \neq b$ shows that all $\varepsilon_i - \varepsilon_j$'s are connected to the $\varepsilon_i + \varepsilon_j$'s (and hence among themselves).

The $-\varepsilon_i - \varepsilon_j$'s are connected to $\varepsilon_i + \varepsilon_j$'s by obvious chains $-\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$.

This shows that any two roots are connected, and Δ is indecomposable. □

Proposition 15.1. *If Δ is decomposable, i.e. $\Delta = \Delta' \cup \Delta''$, with $\alpha' + \alpha'' \notin \Delta \cup \{0\}$ for all $\alpha' \in \Delta, \alpha'' \in \Delta$ then we have a corresponding decomposition of $\mathfrak{g}$ into direct sum of ideals $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$ where*

$$\mathfrak{g}' = \mathfrak{h}' \oplus \left(\bigoplus_{\alpha' \in \Delta'} \mathfrak{g}_{\alpha'} \right)$$

for $\mathfrak{h}' = \text{span}\{\nu^{-1}(\Delta')\}$ and

$$\mathfrak{g}'' = \mathfrak{h}'' \oplus \left(\bigoplus_{\alpha'' \in \Delta''} \mathfrak{g}_{\alpha''} \right)$$

for $\mathfrak{h}'' = \text{span}\{\nu^{-1}(\Delta'')\}$.

Proof. Because $\alpha' + \alpha'' \notin \Delta \cup \{0\}$, we get, $[\mathfrak{g}_{\alpha'}, \mathfrak{g}_{\alpha''}] \subset \mathfrak{g}_{\alpha'+\alpha''} = 0$.

Since $\mathbb{F}[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{F}\nu^{-1}(\Delta')$ we conclude, say by Jacobi identity, $[\mathfrak{h}', \mathfrak{g}_{\alpha''}] = 0$ and $[\mathfrak{h}'', \mathfrak{g}_{\alpha'}] = 0$, so $[\mathfrak{g}', \mathfrak{g}''] = 0$. It remains to show that $\mathfrak{g}'$ and $\mathfrak{g}''$ are subalgebras. First note that $\beta' \in \Delta'$ implies $-\beta' \in \Delta'$. Indeed, we have $-\beta' \in \Delta$, and supposing $-\beta' \in \Delta''$ leads to $\beta' + (-\beta') \in \Delta \cup \{0\}$, a contradiction. Now if $\alpha', \beta' \in \Delta'$ then $\gamma = \alpha' + \beta' \notin \Delta''$, because that would mean $\gamma + (-\beta') = \alpha' \in \Delta$ for $\gamma \in \Delta''$ and $-\beta' \in \Delta'$, another contradiction. So either $\alpha', \beta' \in \Delta' \cup 0$ and $[\mathfrak{g}_{\alpha'}, \mathfrak{g}_{\beta'}] = \mathfrak{g}_{\alpha'+\beta'} \in \mathfrak{g}'$ or $\alpha', \beta' \notin \Delta \cup 0$ and $[\mathfrak{g}_{\alpha'}, \mathfrak{g}_{\beta'}] = 0$. It is now clear that $\mathfrak{g}'$ is a subalgebra. Similarly, $\mathfrak{g}''$ is also a subalgebra. □

Definition. An **abstract root system** is a pair (V, Δ), where V is a finite dimensional vector space over $\mathbb{R}$ of dimension r, and Δ is a finite set of vectors in V, such that the following axioms hold

(1) $0 \notin \Delta$, $V = \text{span}\{\Delta\}$
(2) If $\alpha \in \Delta$ then $k\alpha \in \Delta$ iff $k = 1$ or -1 for all $k \in \mathrm{Z}$
(3) (*string property*) If $\alpha, \beta \in \Delta$then $\{\beta + j\alpha | j \in \mathrm{Z}\} \cap (\Delta \cup 0) = \{\beta + j\alpha | -p \leq j \leq q\}$ where $p, q \in \mathrm{Z}_+$ and $p - q = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$.

Elements of Δ are called roots, and rank of the root system is $\text{rank}(V, \Delta) = r$.

In what follows we will often call abstract root systems simply root systems.

Example 15.1. Let $\mathfrak{g}$ be any semisimple Lie algebra over a chosen field $\mathbb{F}$ of characteristic 0, $\Delta \in \mathfrak{h}^*$ the set of roots, $V = (\text{span over } \mathbb{R} \text{ of } \Delta) = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{h}^*_{\mathbb{Q}}$, where $\mathfrak{h}^*_{\mathbb{Q}}$ is the span of Δ over $\mathbb{Q}$, and $(\cdot, \cdot)$ - the bilinear extension of $K|_{\mathfrak{h}_{\mathbb{Q}}}$ (which we know has rational values on $\mathfrak{h}_{\mathbb{Q}}$) to $\mathfrak{h}_{\mathbb{R}}$- the span of Δ over $\mathbb{R}$. Then by Theorems 3 and 4 from preceding lectures (V, Δ) is an abstract root system called $\mathfrak{g}$-root system.

Example 15.2. dim $V = 1, \Delta = \{\alpha, -\alpha\}$

This is the only root system of rank 1, by the first two axioms. The string property obviously holds here.

Definition. An abstract root system is called **indecomposable** if there is no decomposition into a union of two nonempty sets $\Delta = \Delta' \cup \Delta''$ such that $\alpha' + \alpha'' \notin \Delta \cup \{0\}$ for all $\alpha' \in \Delta', \alpha'' \in \Delta''$.

Proposition 15.2. *A root system (V,Δ) is decomposable iff $V = V' \oplus V''$ where $V' \perp V''$ and $\Delta = \Delta' \cup \Delta''$ where $\Delta' = V' \cap \Delta$ and $\Delta'' = V'' \cap \Delta$.*

Proof. First, suppose that $\Delta = \Delta' \cup \Delta''$ such that $\alpha' + \alpha'' \notin \Delta \cup \{0\}$ for all $\alpha' \in \Delta', \alpha'' \in \Delta''$. Then by the string property $q = 0$. But $-\alpha'' \in \Delta''$ (since if $-\alpha'' \in \Delta'$ then $\alpha'' + (-\alpha'') = 0 \in \Delta \cup \{0\}$, a contradiction), hence $\alpha' - \alpha'' \notin \Delta \cup \{0\}$, so, by the string property $p = 0$ as well, and hence $(\alpha', \alpha'') = 0$. Taking $V' = \operatorname{span}\Delta'$, $V'' = \operatorname{span}\Delta''$ we get the desired decomposition of V.

Conversely, suppose $V = V' \oplus V''$ and $\Delta = \Delta' \cup \Delta''$ for $\Delta' = V' \cap \Delta$, $\Delta'' = V'' \cap \Delta$ and $(\alpha', \alpha'') = 0$ for all $\alpha' \in \Delta'$, $\alpha'' \in \Delta''$. We shall show that $\alpha' + \alpha'' \notin \Delta \cup \{0\}$ for all $\alpha' \in \Delta', \alpha'' \in \Delta''$, so that $\Delta = \Delta' \cup \Delta''$ is a decomposition of Δ. Note that $\alpha' + \alpha'' \neq 0$ because $\alpha' \in V'$, $\alpha'' \in V''$ and $V' \cap V'' = 0$. If $\alpha' + \alpha'' \in \Delta$ then either $\alpha' + \alpha'' \in \Delta'$ or Δ''. Without loss of generality, suppose $\alpha' + \alpha'' \in \Delta''$. But then $0 = (\alpha', \alpha' + \alpha'') = (\alpha', \alpha')$, which is impossible. □

Conclusion. Thus (V,Δ) uniquely decomposes into a sum of indecomposable root systems $V = \oplus_j V_j$, where $V_i \perp V_j$ for $i \neq j$, and $\Delta = \cup_j \Delta_j$ with $\Delta_j \subset V_j$.

This reduces the study of general root systems to indecomposable ones.

Definition. Two indecomposable abstract root systems (V_1, Δ_1) an (V_2, Δ_2) are **isomorphic** if there exists a vector space isomorphism $\varphi : V_1 \mapsto V_2$ such that $\varphi(\Delta_1) = \Delta_2$ and $(\varphi(a), \varphi(b)) = \gamma(a, b)$ for all $a, b \in V_1$ and a constant $\gamma \in \mathbb{R}^+$.

Note. If $(\cdot,\cdot)$ is replaced with $\gamma(\cdot,\cdot)$ for some $\gamma \in \mathbb{R}^+$ we get an isomorphic root system.

Proposition 15.3. *If (V,Δ) is an indecomposable abstract root system, and $(\cdot,\cdot)$ its bilinear form, then for any other positive definite symmetric bilinear form $(\cdot,\cdot)_1$ for which the string property (the third axiom) holds, there exists a $\delta \in \mathbb{R}^+$, such that $(\alpha,\beta)_1 = \delta(\alpha,\beta)$.*

Proof. Fix $\alpha \in \Delta$. Then for any $\beta \in \Delta$ there exist a sequence $\alpha = \gamma_0, \gamma_1, \dots, \gamma_k = \beta$ such that $\gamma_i + \gamma_{i+1} \in \Delta \cup \{0\}$ and $(\gamma_i, \gamma_{i+1}) \neq 0$ for all $i = 0, \dots, k-1$.

To see that, for a root α define $C = \{\gamma \in \Delta |$ there exists a sequence of γ's as above $\}$ and $B = \Delta \setminus C$. Then $\gamma \in C$ and $\beta \in B$ imply $(\gamma, \beta) = 0$. Indeed, $(\gamma,\beta)\neg 0$ by string property means either $\gamma + \beta \in \Delta$ or $\gamma - \beta \in \Delta$, and so the strings γ, β or $\gamma, -\beta, \beta$, respectively, imply $\beta \in C$, a contradiction. Now C and B are orthogonal subsets of Δ and C is nonempty, since it contains α. By proposition 15.2, and since Δ is indecomposable, we conclude that B is empty, as wanted.

Now, define δ by $(\alpha,\alpha)_1 = \delta(\alpha,\alpha)$. We will show that $(\beta,\beta)_1 = \delta(\beta,\beta)$ for the same δ. We know

$$\frac{2(\alpha,\gamma_1)}{(\alpha,\alpha)} = \frac{2(\alpha,\gamma_1)_1}{(\alpha,\alpha)_1} = p - q$$

so $(\alpha,\gamma_1) = \delta(\alpha,\gamma_1)$, and as

$$\frac{2(\alpha,\gamma_1)}{(\gamma_1,\gamma_1)} = \frac{2(\alpha,\gamma_1)_1}{(\gamma_1,\gamma)_1} = p - q$$

and $(\alpha,\gamma_1) \neq 0$ we get $(\gamma_1,\gamma_1)_1 = \delta(\gamma_1,\gamma_1)$.

Similarly $(\gamma_2,\gamma_2)_1 = \delta(\gamma_2,\gamma_2)$ and so on, until $\gamma_k = \beta$.

Finally the string property implies $(\alpha,\beta)_1 = \delta(\alpha,\beta)$ for all roots, and as the roots span V the same is true for all a, b. □

We have 4 series of abstract root systems:

Type A_r: We take $V = \{a \in \mathbb{R}^{r+1} \mid \Sigma a_i = 0\}$ and for the standard basis ε_i of $\mathbb{R}^{r+1}$ we let $\Delta_{A_r} = \{\varepsilon_i - \varepsilon_j \mid i, j = 1, \dots, r+1\}$. The inner product is the standart one on $\mathbb{R}^{n+1}$ i.e. $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$. To check that this is a root system we only need to verify the string property (the other two axioms obviously hold). The explicit computation for $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = \varepsilon_k - \varepsilon_l$ is given by the following cases (note that $(\alpha,\alpha) = 2$ for all roots α):

(1) $i \neq k, l; j \neq k, l$. Then $(\alpha,\beta) = 0$ and $\alpha + \beta, \alpha - \beta \notin \Delta \cup \{0\}$, so $p = q = 0$ and the string property holds.
(2) $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = \varepsilon_j - \varepsilon_k$. Then $(\alpha,\beta) = -1$ and $q = 1$, $p = 0$, so the string property holds again.
(3) $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = -\varepsilon_i + \varepsilon_k$. Then $(\alpha,\beta) = -1$ and $q = 1$, $p = 0$, so the string property holds.

(4) $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = -\varepsilon_i - \varepsilon_j$. Then $(\alpha, \beta) = -2$ and $q = 2$, $p = 0$, so the string property holds.
(5) $\alpha = \varepsilon_i - \varepsilon_j = \beta$. Then $(\alpha, \beta) = 2$ and $q = 0$, $p = 2$, so the string property holds.

Exercise 2. *Taking $V = \mathbb{R}^r$ with the standard basis ε_i, $i = 1, \dots r$ we define*

$$\Delta_{B_r} = \Delta_{so_{2r+1}} = \{\varepsilon_i - \varepsilon_j, -\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j; \varepsilon_i, -\varepsilon_i \mid i, j = 1, \dots, r$$

$$\Delta_{C_r} = \Delta_{sp_{2r}} = \{\varepsilon_i - \varepsilon_j, -\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j; 2\varepsilon_i, -2\varepsilon_i \mid i, j = 1, \dots, r\}$$

$$\Delta_{D_r} = \Delta_{so_{2r}} = \{\varepsilon_i - \varepsilon_j, -\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \mid i, j = 1, \dots, r\}$$

Prove that these are root systems, i.e. that the string property holds by explicitly *verifying it.*

Solution. We have the following obvious

Observation. Suppose a pair (V, Δ) of a finite dimensional vector space over $\mathbb{R}$ and a finite set of vectors in it satisfy the first two axioms of abstract root system, and $\alpha, \beta \in \Delta$. Then the string property holds for α, β iff it holds for $\alpha, -\beta$ iff it holds for $-\alpha, \beta$.

We will carry out the proof for all three of above root systems in parallel. We have the following cases:

0, 0: $\alpha = \varepsilon_i - \varepsilon_j, \beta = \varepsilon_k - \varepsilon_l$. In all three root systems,
(1) $i \neq k, l; j \neq k, l$. Then $(\alpha, \beta) = 0$ and $\alpha + \beta, \alpha - \beta \notin \Delta \cup \{0\}$ so $p = q = 0$ and the string property holds.
(2) $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = \varepsilon_j - \varepsilon_k$. Then $(\alpha, \beta) = -1$ and $q = 1$, $p = 0$, so the string property holds again.
(3) $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = -\varepsilon_i + \varepsilon_k$. Then $(\alpha, \beta) = -1$ and $q = 1$, $p = 0$, so the string property holds.
(4) $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = -\varepsilon_i - \varepsilon_j$. Then $\alpha = -\beta$ and automatically $q = 2$, $p = 0$, so the string property holds.
(5) $\alpha = \varepsilon_i - \varepsilon_j = \beta$. Then automatically $q = 0$, $p = 2$, so the string property holds.

0, 1: $\alpha = \varepsilon_i - \varepsilon_j, \beta = \varepsilon_k + \varepsilon_l$.
(1) $i \neq k, l; j \neq k, l$. Then $(\alpha, \beta) = 0$ and $p = q = 0$ for all three root systems, the string property holds.
(2) $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = \varepsilon_j + \varepsilon_k$. Then $(\alpha, \beta) = -1$ and $q = 1$, $p = 0$ for all three root systems, the string property holds.
(3) $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = \varepsilon_i + \varepsilon_k$. Then $(\alpha, \beta) = 1$ and $q = 0$, $p = 1$,for all three root systems, the string property holds.
(4) $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = \varepsilon_i + \varepsilon_j$. Then $(\alpha, \beta) = 0$ and $q = p = 1$ for Δ_{C_r}, $p = q = 0$ for $\Delta_{B_r}, \Delta_{D_r}$,and the string property holds in all three.

1, 0: $\alpha = \varepsilon_i + \varepsilon_j, \beta = \varepsilon_k - \varepsilon_l$.
(1) $i \neq k, l; j \neq k, l$. Then $(\alpha, \beta) = 0$ and $p = q = 0$ for all three root systems, the string property holds.
(2) $\alpha = \varepsilon_i + \varepsilon_j$, $\beta = -\varepsilon_j + \varepsilon_k$. Then $(\alpha, \beta) = -1$ and $q = 1$, $p = 0$ for all three root systems, the string property holds.
(3) $\alpha = \varepsilon_i + \varepsilon_j$, $\beta = \varepsilon_i - \varepsilon_k$. Then $(\alpha, \beta) = 1$ and $q = 0$, $p = 1$,for all three root systems, the string property holds.
(4) $\alpha = \varepsilon_i + \varepsilon_j$, $\beta = \varepsilon_i - \varepsilon_j$. Then $(\alpha, \beta) = 0$ and $q = p = 1$ for Δ_{C_r}, $p = q = 0$ for $\Delta_{B_r}, \Delta_{D_r}$,and the string property holds in all three.

1, 1: $\alpha = \varepsilon_i + \varepsilon_j, \beta = \varepsilon_k + \varepsilon_l$. Then in all three root systems
(1) $i \neq k, l; j \neq k, l$. Then $(\alpha, \beta) = 0$ and $p = q = 0$, the string property holds.
(2) $\alpha = \varepsilon_i + \varepsilon_j$, $\beta = \varepsilon_j + \varepsilon_k$. Then $(\alpha, \beta) = 1$ and $q = 0$, $p = 1$, the string property holds.
(3) $\alpha = \varepsilon_i + \varepsilon_j = \beta$ Then automatically $q = 0$, $p = 2$, so the string property holds.

By the Observation we get that the string property holds in all three root systems in the cases

0,-1: $\alpha = \varepsilon_i - \varepsilon_j, \beta = -\varepsilon_k - \varepsilon_l$

-1, 0: $\alpha = -\varepsilon_i - \varepsilon_j, \beta = \varepsilon_k - \varepsilon_l$
-1, 1: $\alpha = -\varepsilon_i - \varepsilon_j, \beta = \varepsilon_k + \varepsilon_l$
1,-1: $\alpha = \varepsilon_i + \varepsilon_j, \beta = -\varepsilon_k - \varepsilon_l$
-1,-1: $\alpha = e_i - \varepsilon_j, \beta = -\varepsilon_k - \varepsilon_l$

This completes the proof for Δ_{D_r} and leaves only the cases involving "singletons" for the other two root systems.

⋄, 0: $\alpha = (2)\varepsilon_i$, $\beta = \varepsilon_j - e_k$.
(1) $i \neq j, k$. Then $(\alpha, \beta) = 0$ and $p = q = 0$, the string property holds.
(2) $i = j$. Then $(\alpha, \beta) = 1$ for Δ_{B_r} and $q = 0$, $p = 2$, the string property holds and $(\alpha, \beta) = 2$, $(\alpha, \alpha) = 4$ for Δ_{B_r} and $q = 0$, $p = 1$, the string property holds.
(3) $i = k$. Then $(\alpha, \beta) = -1$ for Δ_{B_r} and $q = 2$, $p = 0$, the string property holds and $(\alpha, \beta) = -2$, $(\alpha, \alpha) = 4$ for Δ_{C_r} and $q = 0$, $p = 2$, the string property holds.

⋄, 1: $\alpha = (2)\varepsilon_i$, $\beta = \varepsilon_j + e_k$.
(1) $i \neq j, k$. Then $(\alpha, \beta) = 0$ and $p = q = 0$, the string property holds.
(2) $i = j$. Then $(\alpha, \beta) = 1$ for Δ_{B_r} and $q = 0$, $p = 2$, the string property holds and $(\alpha, \beta) = 2$, $(\alpha, \alpha) = 4$ for Δ_{C_r} and $q = 0$, $p = 1$, the string property holds.

0, ⋄: $\alpha = \varepsilon_j - e_k$, $\beta = (2)\varepsilon_i$.
(1) $i \neq j, k$. Then $(\alpha, \beta) = 0$ and $p = q = 0$, the string property holds.
(2) $i = j$. Then $(\alpha, \beta) = 1$ for Δ_{B_r} and $q = 0$, $p = 1$, the string property holds and $(\alpha, \beta) = 2$ for Δ_{B_r} and $q = 0$, $p = 2$, the string property holds.
(3) $i = k$. Then $(\alpha, \beta) = -1$ for Δ_{B_r} and $q = 1$, $p = 0$, the string property holds and $(\alpha, \beta) = -2$ for Δ_{B_r} and $q = 2$, $p = 1$, the string property holds.

1, ⋄: $\alpha = \varepsilon_i + e_j$, $\beta = (2)\varepsilon_k$.
(1) $i \neq j, k$. Then $(\alpha, \beta) = 0$ and $p = q = 0$, the string property holds.
(2) $i = j$. Then $(\alpha, \beta) = 1$ for Δ_{B_r} and $q = 0$, $p = 2$, the string property holds and $(\alpha, \beta) = 2$ for Δ_{C_r} and $q = 0$, $p = 2$, the string property holds.

⋄,⋄: $\alpha = (2)\varepsilon_i$, $\beta = (2)\varepsilon_j$.
(1) $i \neq j$. Then $(\alpha, \beta) = 0$ and $p = q = 1$ in both, the string property holds.
(2) $i = j$ This is automatic.

By the Observation we get that the string property holds in both root systems in the cases
-⋄, 0: $\alpha = -(2)\varepsilon_i$, $\beta = \varepsilon_j - e_k$
-⋄, 1: $\alpha = -(2)\varepsilon_i$, $\beta = \varepsilon_j + e_k$
⋄,-1: $\alpha = (2)\varepsilon_i$, $\beta = -\varepsilon_j - e_k$
-⋄,-1: $\alpha = -(2)\varepsilon_i$, $\beta = -e_j - e_k$
0,-⋄: $\alpha = \varepsilon_j - e_k$, $\beta = -(2)\varepsilon_i$
-1, ⋄: $\alpha = -\varepsilon_i - e_j$, $\beta = (2)\varepsilon_k$
1,-⋄: $\alpha = \varepsilon_i + e_j$, $\beta = -(2)\varepsilon_k$
-1,-⋄: $\alpha = -\varepsilon_i - e_j$, $\beta = -(2)\varepsilon_k$
-⋄, ⋄: $\alpha = -(2)\varepsilon_i$, $\beta = (2)\varepsilon_j$
⋄,-⋄: $\alpha = (2)\varepsilon_i$, $\beta = -(2)\varepsilon_j$
-⋄,-⋄: $\alpha = -(2)\varepsilon_i$, $\beta = -(2)\varepsilon_j$

This exhausts the possibilities, and shows that Δ_{B_r} and Δ_{C_r} are indeed root systems.

□