18.745 Introduction to Lie Algebras Fall 2004

Lecture 16: More On Root Systems

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The exercises will be stated as they occurred during the course of the lecture, embedded within the rest of the notes. Then, they will be restated at the end of all the notes with solutions.

Remark We proved previously that the root system (V, Δ) coming from a finite dimensional Lie algebra is of the form $V = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{h}^*_{\mathbb{Q}}$ and $(\alpha, \beta) \in \mathbb{Q}$ for $\alpha, \beta \in \Delta$. Also, choosing basis $B = \{\beta_1, \beta_2, \ldots, \beta_r\}$ of V consisting of roots, any other root is a linear combination of elements from B with rational coefficients.

Indeed, if $\alpha = \sum_i c_i \beta_i$, then $(\alpha, \beta_j) = \sum_i c_i(\beta_i, \beta_j)$, where $[(\beta_i, \beta_j)]^r_{i,j=1}$ is a non-singular matrix over $\mathbb{Q}$, and hence the c_i are rational numbers by Cramer's Rule.

As for indecomposable abstract root systems, it follows from the proof of uniqueness of $(\cdot, \cdot)$ up to a factor that, after multiplying it by a factor so that $(\alpha, \alpha) \in \mathbb{Q}$ for one root α, we will have $(\alpha, \beta) \in \mathbb{Q}$ for all $\alpha, \beta \in Q$ and the above argument again applies.

Example $1, \sqrt{2}$ are linearly independent over $\mathbb{Q}$ but are dependent over $\mathbb{R}$. Also, the set $\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \sqrt{2}$ is not discrete in $\mathbb{R}$.

Definition A lattice L in a Euclidean space V over $\mathbb{R}$ is a discrete subgroup of V which spans V over $\mathbb{R}$.

Example Let Δ be a finite subset of V such that $(\alpha, \beta) \in \mathbb{Q}$ for $\forall \alpha, \beta \in \Delta$. Then, $\mathbb{Z}\,\Delta$ is a lattice in V. Indeed, choose a basis $B \subset \Delta$ of V. All vectors of Δ will have rational coordinates in the basis, and hence $\mathbb{Z}\,\Delta$ is a discrete subgroup of V.

Equivalently, a lattice L in a Euclidean space V over $\mathbb{R}$ of dimension r is a free abelian subgroup of rank r, that is $L = \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_r$, where $\beta_1, \ldots, \beta_r$ are linearly independent. The root lattice of an abstract room system (V, Δ) is $Q = \mathbb{Z}\,\Delta$. We have constructed the following root systems:

- $\Delta_{A_r} = \{\epsilon_i - \epsilon_j | i, j = 1, \ldots, r+1,\ i \neq j\}$ and $Q_{A_r} = \{\sum_{i=1}^{r+1} a_i \epsilon_i | \sum_{i=1}^{r+1} a_i = 0 \text{ for } a_i \in \mathbb{Z}\}$ in the vector space $V = \{\sum a_i \epsilon_i | a_i \in \mathbb{R}, \sum a_i = 0\}$.
- $\Delta_{B_r} = \{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i | i, j = 1, \ldots, r,\ i \neq j\}$ and $Q_{B_r} = \{\sum_{i=1}^{r} a_i \epsilon_i | a_i \in \mathbb{Z}\}$ in the vector space $V = \sum \mathbb{R}\epsilon_i$.
- $\Delta_{C_r} = \{\pm\epsilon_i \pm \epsilon_j, \pm 2\epsilon_i | i, j = 1, \ldots, r,\ i \neq j\}$ and $Q_{C_r} = \{\sum_{i=1}^{r} a_i \epsilon_i | a_i \in \mathbb{Z},\ \sum_{i=1}^{r} a_i \in 2\mathbb{Z}\}$ in the vector space $V = \sum \mathbb{R}\epsilon_i$.
- $\Delta_{D_r} = \{\pm\epsilon_i \pm \epsilon_j | i, j = 1, \ldots, r,\ i \neq j\}$ and $Q_{D_r} = Q_{C_r}$ in the vector space $V = \sum \mathbb{R}\epsilon_i$. Here $r \neq 2$.

Exercise Prove that Q_{A_r} and Q_{C_r} are as above.

In addition to the lattices $Q_{A_r}, Q_{B_r}, Q_{C_r}, Q_{D_r}$, there are exceptional lattices. For instance, $\Gamma_r = \{\sum_{i=1}^r a_i \epsilon_i \,|\, \text{all } a_i \in \mathbb{Z} \text{ or all } a_i \in \mathbb{Z}+\frac{1}{2},\ \sum_{i=1}^r a_i \in 2\mathbb{Z}\,\}$. This lives in the vector space $V = \bigoplus_{i=1}^r \mathbb{R}\epsilon_i$ with $(\epsilon_i, \epsilon_j) = \delta_{ij}$. The most important example of such a lattice is Γ_8.

Definition A lattice is called *integral* (resp. *even*) if $(\alpha, \beta) \in \mathbb{Z}$ (resp. $(\alpha, \alpha) \in 2\mathbb{Z}$) for all $\alpha, \beta \in Q$.

Obviously an even lattice is integral: $(\alpha+\beta, \alpha+\beta) = (\alpha, \alpha) + 2(\alpha, \beta) + (\beta, \beta) \to (\alpha, \beta) \in \mathbb{Z}$, because $(\alpha+\beta, \alpha+\beta), (\alpha, \alpha), (\beta, \beta) \in 2\mathbb{Z}$.

Example Γ_{8n} $(n = 1, 2, \ldots)$ are even lattices, since $(\alpha, \alpha) = \sum_{i=1}^n a_i^2 = \sum_{i=1}^n a_i \mod 2$. Recall that $a_i^2 \equiv a_i \mod 2$ if a_i is integral. All other vectors are of the form $\rho + \alpha$ where $\rho = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ and α has integer coordinates. But then $(\rho+\alpha, \rho+\alpha) = (\rho, \rho) + \sum_{i=1}^n a_i + (\alpha, \alpha) = 8n/4 +$ even integer $\in 2\mathbb{Z}$.

Theorem Let Q be an even lattice in a Euclidean space V. Let $\Delta = \{\alpha \in Q \,|\, (\alpha, \alpha) = 2\}$ and assume Δ spans V over $\mathbb{R}$. Then (V, Δ) is an abstract root system.

Proof We show that each of the requirements for an abstract root system is met.

- Δ spans $V \to$ assumed
- Δ is finite $\to$ Δ is intersection of discrete Q and compact sphere $\{v \in V \,|\, (v, v) = 2\}$
- $\alpha \in \Delta \to k\alpha \in \Delta$ if $k = \pm 1$. This is obvious.

It remains to check the string property. Take $\alpha, \beta \in \Delta$. Then, $(\alpha \pm \beta, \alpha \pm \beta) = 4 \pm 2(\alpha, \beta) \geq 0$. Hence, $\alpha + \beta$ is a root iff $(\alpha, \beta) = -1$, and $\alpha - \beta$ is a root iff $(\alpha, \beta) = +1$. $\blacksquare$

All possible inner products (α, β) for $\alpha, \beta \in \Delta$ are:

- +2 iff $\alpha = +\beta$ $(p = 2, q = 0)$
- -2 iff $\alpha = -\beta$ $(p = 0, q = 2)$
- +1 iff $\alpha - \beta \in \Delta$, $\alpha + \beta \notin \Delta$ $(p = 1, q = 0)$
- -1 iff $\alpha + \beta \in \Delta$, $\alpha - \beta \notin \Delta$ $(p = 0, q = 1)$
- 0 iff $\alpha + \beta, \alpha - \beta \notin \Delta$ $(p = q = 0)$

Hence the string property, $p - q = (\alpha, \beta)$, holds in all cases. $\blacksquare$ Let $\Delta_{E_8} = \{\alpha \in \Gamma_8 | (\alpha, \alpha) = 2\} = \{\pm\epsilon_i \pm \epsilon_j (i \neq j), \frac{1}{2}(\pm\epsilon_1 \cdots \pm \epsilon_8)(\text{even \# of - signs})\}$. Then (V, Δ_{E_8}) is an abstract root system. Moreover, $\mathbb{Z}\,\Delta_{E_8} = \Gamma_8$ so $\Gamma_8 = Q_{E_8}$.

Exercise Show that $\mathbb{Z}\,\Delta_{E_8} = \Gamma_8 = Q_{E_8}$.

There are 240 roots in Δ_{E_8}. This makes it larger than Δ_{A_8}, Δ_{B_8}, Δ_{C_8}, and Δ_{D_8}. Respectively, these have $8 \cdot (8+1) = 72$, $2 \cdot 8^2 = 128$, $2 \cdot 8^2 = 128$ and $2 \cdot (8^2 - 8) = 112$ roots.

Exercise Confirm the sizes of Δ_{A_8} through Δ_{D_8}.

The case Γ_8 is the largest exception to the otherwise exhaustive classification into A_r through D_r.

Example Define the following as a sublattice of Q_{E_8}: $Q_{E_7} = \{\alpha \in \Gamma_{E_8} \,|\, (\rho, \alpha) = 0\}$, where $\rho = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ and $\Delta_{E_7} = \{\alpha \in Q_{E_7} \,|\, (\alpha, \alpha) = 2\}$. This is another exceptional algebra that does not fit into the A_r through D_r classification scheme.

Example Define the following as a sublattice of Q_{E_7}: $Q_{E_6} = \{\alpha \in Q_{E_7} \,|\, (\epsilon_7 + \epsilon_8, \alpha) = 0\}$, $\Delta_{E_6} = \{\alpha \in Q_{E_6} \,|\, (\alpha, \alpha) = 2\}$. This is a third exceptional algebra that does not fit into the A_r through D_r classification scheme.

Exercise Find Δ_{E_7} and Δ_{E_6}. Show they respectively span Q_{E_7} and Q_{E_6} over $\mathbb{Z}$. Also, show that $|\Delta_{E_7}| = 126$ and $|\Delta_{E_6}| = 72$.

Another important lattice is $Q_{F_4} = \{\sum_{i=1}^{4} a_i \epsilon_i \,|\, \text{all } a_i \in \mathbb{Z} \text{ or all } a_i \in \mathbb{Z} + \frac{1}{2}\}$. Let $\Delta_{F_4} = \{\alpha \in Q_{F_4} \,|\, (\alpha, \alpha) = 2 \text{ or } 1\} = \{\pm\epsilon_i \pm \epsilon_j \,(i \neq j),\ \pm\epsilon_i,\ \frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\}$. The vector space is $V = \bigoplus_{i=1}^{4} \mathbb{R}\epsilon_i$ with $(\epsilon_i, \epsilon_j) = \delta_{ij}$.

Exercise Check that Δ_{F_4} is an abstract root system and that it spans Q_{F_4}. Also check that $|\Delta_{F_4}| = 48$.

The last exceptional root system is $\Delta_{G_2} = \{\alpha \in Q_{A_2} \,|\, (\alpha, \alpha) = 2 \text{ or } 6\} = \{\epsilon_i - \epsilon_j \,(i, j = 1, 2, 3;\ i \neq j),\ \pm(\epsilon_i + \epsilon_j - 2\epsilon_k)\,(i \neq j \neq k \neq i)\}$. We can calculate that $|\Delta_{G_2}| = 12$.

Exercise Check that the string property holds for Δ_{G_2}.

There are three distinct indecomposable root systems (V, Δ) that live in a vector space V with dimension 2. These are $A_2 = \{\epsilon_i - \epsilon_j \,(i, j = 1, 2, 3;\ i \neq j)\}$, $B_2 = \{\pm\epsilon_i \pm \epsilon_j,\ \pm\epsilon_i \,(i \neq j;\ i, j = 1, 2)\} \simeq C_2$ and G_2 as defined above. They are depicted below.

1 Exercises and Solutions

Exercise Prove that Q_{A_r} and Q_{C_r} are as above.

Solution First the solution regarding Q_{A_r}, and then that regarding Q_{C_r}.

Claim $Q_{A_r} \subset \mathbb{Z}\,\Delta_{A_r}$

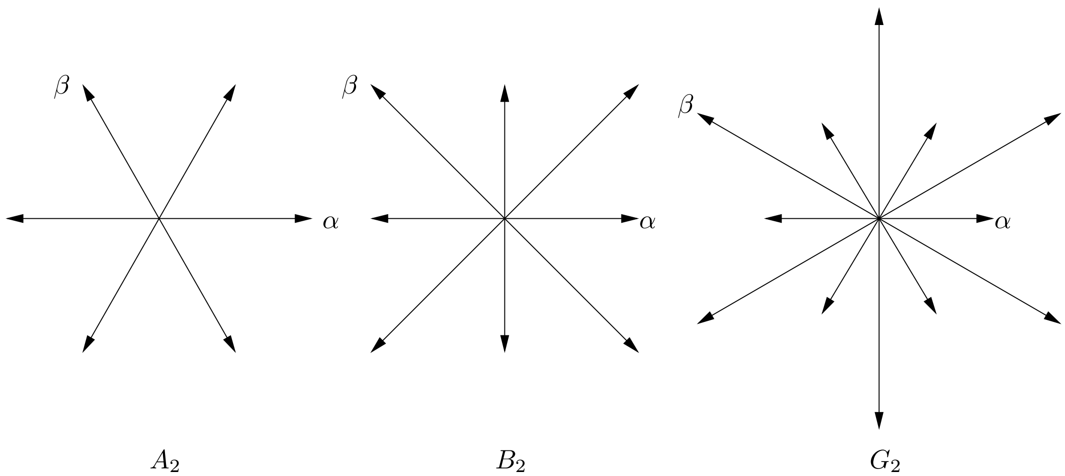

Figure 1: Here $\{\alpha, \beta\}$ is a subset of the simple roots.

Proof Go through the list for $i = 1, \ldots, r+1$, attributing $a_i\epsilon_i$ to $-a_i\epsilon_{i+1}$. That is:

$$
\begin{aligned}
&a_1\epsilon_1 \to a_1(\epsilon_1 - \epsilon_2) + a_1\epsilon_2 \\
&a_2\epsilon_2 + a_1\epsilon_2 \to (a_1 + a_2)(\epsilon_2 - \epsilon_3) + (a_1 + a_2)\epsilon_3 \\
&(a_1 + a_2 + a_3)\epsilon_3 \to (a_1 + a_2 + a_3)(\epsilon_3 - \epsilon_4) + (a_1 + a_2 + a_3)\epsilon_4 \\
&\qquad\vdots \\
&(a_1 + \cdots + a_r)\epsilon_r \to (a_1 + \cdots a_r)(\epsilon_r - \epsilon_{r+1} + (a_1 + \cdots + a_r)\epsilon_{r+1} = (a_1 + \cdots a_r)(\epsilon_r - \epsilon_{r+1} \\
&(a_1 + \cdots + a_{r+1})\epsilon_r = 0 \cdot \epsilon_r \qquad ∎
\end{aligned}
$$

Claim $\mathbb{Z}\,\Delta_{A_r} \subset Q_{A_r}$

Proof Each $\alpha \in \mathbb{Z}\Delta_{A_r}$ is of the form $a(\epsilon_i - \epsilon_j)$ for some $a \in \mathbb{Z}$. Then, any linear combination of such roots come in pairs of ϵ's, one with coefficient $+a$ and the other with $-a$. Thus, the sum of the coefficients is 0 and so is contained in $\mathbb{Z}$. ∎

$\mathbb{Z}\,\Delta_{A_r} \subset Q_{A_r}$ and $Q_{A_r} \subset \mathbb{Z}\,\Delta_{A_r} \Rightarrow \mathbb{Z}\,\Delta_{A_r} = Q_{A_r}$. ∎

Claim $Q_{C_r} \subset \mathbb{Z}\,\Delta_{C_r}$

Proof Use the same process as with the Q_{A_r} proof, getting roots of the form $a_1(\epsilon_1 - \epsilon_2), (a_1 + a_2)(\epsilon_2 - \epsilon_3), \ldots, (a_1 + \cdots + a_r)(\epsilon_r - \epsilon_{r+1})$. We have $(a_1 + \cdots + a_r)\epsilon_{r+1}$ left. By the definition of Q_{C_r}, this sum $\in 2\mathbb{Z}$. This, then, is of the form $\mathbb{Z}\,\Delta_{C_r}$ since $2\epsilon_{r+1} \in \Delta_{C_r}$. ∎

Claim $\mathbb{Z}\,\Delta_{C_r} \subset Q_{C_r}$

Proof Each term $\alpha \in \mathbb{Z}\,\Delta_{C_r}$ is of the form:

- $a\,(+\epsilon_i + \epsilon_j) \to$ contributes $+2$ to sum

- $a\,(-\epsilon_i + \epsilon_j) \rightarrow$ contributes 0 to sum
- $a\,(-\epsilon_i - \epsilon_j) \rightarrow$ contributes -2 to sum
- $2a\epsilon_i \rightarrow$ contributes $+2$ to sum

for $a \in \mathbb{Z}$. Each of these possibilities contribute an even integer to the sum of all the coefficients, showing that $\sum_i a_i \in 2\mathbb{Z}$. ∎

Thus, $\mathbb{Z}\,\Delta_{C_r} \subset Q_{C_r}$ and $Q_{C_r} \subset \mathbb{Z}\,\Delta_{C_r} \Rightarrow Q_{C_r} = \mathbb{Z}\,\Delta_{C_r}$ QED.

Exercise Show that $\mathbb{Z}\,\Delta_{E_8} = \Gamma_8 = Q_{E_8}$.

Solution We will use a similar technique as with the previous exercise. Recall that $\Gamma_8 = \{\sum_{i=1}^{8} a_i\epsilon_i \,|\, \forall a_i \in \mathbb{Z}$ or $\forall a_i \in \mathbb{Z} + \frac{1}{2}$; $\sum_{i=1}^{8} a_i \in 2\mathbb{Z}\}$ and $\Delta_{E_8} = \{\alpha \in \Gamma_8 \,|\, (\alpha, \alpha) = 2\} = \{\pm\epsilon_i \pm \epsilon_j\ (i \neq j);\ \frac{1}{2}(\pm\epsilon_1 \cdots \pm \epsilon_8)$ (even # of - signs)$\}$.

Claim $Q_{E_8} \equiv \mathbb{Z}\,\Delta_{E_8} \subset \Gamma_8$

Proof Clearly both general types of roots in Δ_{E_8} satisfy $\forall a_i \in \mathbb{Z}$ or $\forall a_i \in \mathbb{Z} + \frac{1}{2}$. Now we check the constraint on the sum of the coefficients.

- $+\epsilon_i + \epsilon_j \rightarrow +2\mathbb{Z}$
- $\pm(\epsilon_i - \epsilon_j) \rightarrow 0$
- $-\epsilon_i - \epsilon_j \rightarrow -2\mathbb{Z}$
- $\frac{1}{2}(\pm\epsilon_1 \cdots \pm \epsilon_8) \rightarrow (4, 2, 0, -2, -4)\,\mathbb{Z} \in 2\mathbb{Z}$

Claim $\Gamma_8 \subset \mathbb{Z}\,\Delta_{E_8} \equiv Q_{E_8}$

Proof Separate elements of Γ_8 into two cases, $\forall a_i \in \mathbb{Z}$ and $\forall a_i \in \mathbb{Z} + \frac{1}{2}$, and treat separately.

1. cd Decompose $\sum_{i=1}^{8} a_i\epsilon_i \in \Gamma_8$ into $a_1(\epsilon_1 - \epsilon_2) + (a_1 + a_2)(\epsilon_2 - \epsilon_3) + \cdots + (a_1 + \cdots + a_7)(\epsilon_7 - \epsilon_8) + (a_1 + \cdots + a_8)\epsilon_8$. But we know that $a_1 + \cdots + a_8 \equiv 2S \in 2\mathbb{Z}$. Then, write $2S\epsilon_8 = S(\epsilon_8 - \epsilon_1) + S(\epsilon_8 - \epsilon_2) + S(\epsilon_1 + \epsilon_2)$. We have thus expressed this element of Γ_8 as an element of $\mathbb{Z}\,\Delta_{E_8}$.
2. First, we know that since $\sum_{i=1}^{8} \in 2\mathbb{Z}$, $\sum_{i=1}^{8} a_i\epsilon_i$ can be written as a sum of terms of the form $\frac{1}{2}(\pm\epsilon_1 \cdots \pm \epsilon_8)$. Translate the problem to the "nearest" problem of the former type by sending $a_i \rightarrow a_i + \frac{1}{2}$. Then, we know that $\sum_{i=1}^{8} a_i \in 2\mathbb{Z}$ by the above argument. Since $(-\frac{1}{2}, \ldots, -\frac{1}{2}) \in \Delta_{E_8}$, then $\sum_{i=1}^{8}$ must also $\in \mathbb{Z}\,\Delta_{E_8}$.

Thus, $\Gamma_8 \subset \mathbb{Z}\,\Delta_{E_8}$ and $\mathbb{Z}\,\Delta_{E_8} \subset \Gamma_8 \Rightarrow \mathbb{Z}\,\Delta_{E_8} \subset \Gamma_8$. ∎

Exercise Confirm the sizes of Δ_{A_8} through Δ_{D_8}.

Solution

- Δ_{A_r}: Recall that $\Delta_{A_r} = \{\epsilon_i - \epsilon_j \,|\, i,j = 1,\ldots,r+1,\ i \neq j\}$. There are $\frac{(r+1)!}{(r-1)!\,2!}$ ways of choosing a pair of indices i and j. Each pair has 2 ways of being arranged around the minus sign. Thus, $r(r+1)$ roots exist.
- $\Delta_{B_r} = \{\pm\epsilon_i \pm \epsilon_j;\, \pm\epsilon_i \,|\, i,j = 1,\ldots,r; i \neq j\}$. The first sort of root has 2^2 choices of sign arrangement and $\frac{r(r+1)}{2}$ choices of indices i,j. The second sort of root has 2 choices of sign and r choices of index. This gives a total of $2r(r-1) + 2r = 2r^2$ roots.
- $\Delta_{C_r} = \{\pm\epsilon_i \pm \epsilon_j;\, \pm 2\epsilon_i \,|\, i,j = 1,\ldots,r; i \neq j\}$. Identical to the above case.
- $\Delta_{D_r} = \{\pm\epsilon_i \pm \epsilon_j \,|\, i,j = 1,\ldots,r; i \neq j\}$. This case just has $2r(r-1)$ roots, for reasons stated in the Δ_{B_r} case.

These solutions all hold in the specific case $r = 8$. $\blacksquare$.

Exercise Find Δ_{E_7} and Δ_{E_6}. Show they respectively span Q_{E_7} and Q_{E_6} over $\mathbb{Z}$. Also, show that $|\Delta_{E_7}| = 126$ and $|\Delta_{E_6}| = 72$.

Solution $\Delta_{E_7} = \{\alpha \in Q_{E_7} \,|\, (\alpha,\alpha) = 2\} \subset \Delta_{E_8} = \{\pm\epsilon_i \pm \epsilon_j\ (i \neq j),\ \frac{1}{2}(\pm\epsilon_1 \cdots \pm \epsilon_8)\}$. To get Δ_{E_7} take Δ_{E_8} and impose the condition $(\rho,\alpha) = 0$. Impose the same constraint on Q_{E_8} to obtain Q_{E_7}. Since Δ_{E_8} spanned Q_{E_8}, it is clear that Δ_{E_7} obtained as above will span Q_{E_7} obtained this same way.

$|\Delta_{E_7}| = \frac{8!}{4!\,4!} + 2\frac{8!}{6!\,2!} = 70 + 2\cdot 28 = 126.$

Δ_{E_6} is obtained from Δ_{E_7} by imposing the condition that $(\epsilon_7 + \epsilon_8, \alpha) = 0$. Applying the same condition to Q_{E_7} gives Q_{E_6}. Then, the same argument as above indicates that Δ_{E_6} spans Q_{E_6}.

Actually imposing this condition on the given expression of Δ_{E_6} gives the alternative expression $\Delta_{E_6} = \{\frac{1}{2}(\pm\epsilon_1 \cdots \pm \epsilon_6) \pm \frac{1}{2}(\epsilon_7 - \epsilon_8)$ (with a total of 4 - signs); $\pm(\epsilon_i - \epsilon_j)\,(i,j = 1,\ldots,6; i \neq j)$; $\pm(\epsilon_7 - \epsilon_8)\}$. Then, we calculate $|\Delta_{E_6}| = 2\frac{6!}{3!\,3!} + 2\frac{6!}{2!\,4!} + 2 = 40 + 30 + 2 = 72$.

Exercise Check that Δ_{F_4} is an abstract root system and that it spans Q_{F_4}. Also check that $|\Delta_{F_4}| = 48$.

Solution $\Delta_{F_4} = \{\alpha \in Q_{F_4} \,|\, (\alpha,\alpha) = 2 \text{ or } 1\} = \{\pm\epsilon_i \pm \epsilon_j\ (i \neq j);\ \pm\epsilon_i;\ \frac{1}{2}(\pm\epsilon_1 \cdots \pm \epsilon_4)\}$ and $Q_{F_4} = \{\sum_{i=1}^{4} a_i\epsilon_i \,|\, \forall a_i \in \mathbb{Z} \text{ or } \forall a_i \in \mathbb{Z} + \frac{1}{2}\}$.

1. $|\Delta_{F_4}| = 2^2\frac{4!}{2!\,2!} + 2\frac{4!}{3!\,1!} + 2^4\frac{4!}{4!\,0!} = 48$
2. Roots in Δ_{F_4} of the first two types have integer coefficients. Sums of integers times these roots have only integer coefficients, therefore. If an odd number of roots of the last type with half integer coefficients occur, the overall sum has half-integer coefficients in front of the ϵ_i's. Otherwise, they are still integers. Thus, $Q_{F_4} \subset \mathbb{Z}\,\Delta_{F_4}$.
3. Properties 1 and 2 of an abstract root system obviously satisfied. All that remains is to show that the string property holds.

- $\alpha = \epsilon_i + \epsilon_j,\ \beta = \epsilon_k - \epsilon_m\ (i \neq m,\ j \neq k)$. $\langle \beta, \alpha \rangle = \delta_{ik} - \delta_{jm}$. $p = \delta_{ik}$ and $q = \delta_{jm}$.
- $\alpha = \epsilon_i + \epsilon_j,\ \beta = \epsilon_k$. $\langle \beta, \alpha \rangle = \delta_{ik} + \delta_{jk}$. $p = \delta_{ik} + \delta_{jk}$ and $q = 0$.
- $\alpha = \epsilon_i,\ \beta = \epsilon_j + \epsilon_k$. $\langle \beta, \alpha \rangle = 2\delta_{ij} + 2\delta_{ik}$. $p = 2\delta_{ij} + 2\delta_{ik}$ and $q = 0$.
- $\alpha = \epsilon_i,\ \beta = \epsilon_j - \epsilon_k$. $\langle \beta, \alpha \rangle = 2\delta_{ij} - 2\delta_{ik}$. $p = 2\delta_{ij}$ and $q = 2\delta_{ik}$.
- $\alpha = \epsilon_i,\ \beta = \frac{1}{2}(\pm\epsilon_1 \cdots \pm\epsilon_4)$. $(\beta, \alpha) = \pm\frac{1}{2}$ (+ if $+\epsilon_i$ in α and - if $-\epsilon_i$ in α) $\Rightarrow \langle \beta, \alpha \rangle = \pm 1$ with same conditions. $p = 1$(if $+\epsilon_i$), 0(if $-\epsilon_i$) and $q = 1$(if -ϵ_i), 0(if $+\epsilon_i$).
- $\alpha = \frac{1}{2}(\pm\epsilon_1 \cdots \pm \epsilon_4),\ \beta = \epsilon_i + \epsilon_j$. $\langle \beta, \alpha \rangle = \mathrm{sign}_\alpha(\epsilon_i) + \mathrm{sign}_\alpha(\epsilon_j)$. $p = 1$(if $+\epsilon_i + \epsilon_j$ in α, 0(else)) and $q = 1$(if $-\epsilon_i - \epsilon_j$ in α), 0(else).
- $\alpha = \frac{1}{2}(\pm\epsilon_1 \cdots \pm \epsilon_4),\ \beta = \epsilon_i$. $\langle \beta, \alpha \rangle = \mathrm{sign}_\alpha(\epsilon_i)$. $p = 1$(if $+\epsilon_i$ in α), 0(else) and $q = 1$(if $-\epsilon_i$ in α), 0(else).

Exercise Check that the string property holds for Δ_{G_2}.

Solution $\Delta_{G_2} = \{\alpha \in Q_{A_2} \,|\, (\alpha, \alpha) = 2 \text{ or } 6\} = \{\epsilon_i - \epsilon_j\ (i, j = 1, 2, 3\ i \neq j);\ \pm(\epsilon_i + \epsilon_j - 2\epsilon_k)\ (i \neq j \neq k \neq i)\}$

- $\alpha = \epsilon_i - \epsilon_j,\ \beta = \epsilon_i - \epsilon_k \to \langle \beta, \alpha \rangle = 1$. $p = 2, q = 1$.
- $\alpha = \epsilon_i - \epsilon_j,\ \beta = \epsilon_k - \epsilon_k \to \langle \beta, \alpha \rangle = 1$. $p = 2, q = 1$.
- $\alpha = \epsilon_i - \epsilon_j,\ \beta = \epsilon_i + \epsilon_j - 2\epsilon_k \to \langle \beta, \alpha \rangle = 0$. $p = 0, q = 0$.
- $\alpha = \epsilon_i - \epsilon_j,\ \beta = \epsilon_i + \epsilon_k - 2\epsilon_j \to \langle \beta, \alpha \rangle = 3$. $p = 3, q = 0$.
- $\alpha = \epsilon_i + \epsilon_j - 2\epsilon_k,\ \beta = \epsilon_i + \epsilon_k - 2\epsilon_j \to \langle \beta, \alpha \rangle = -1$. $p = 0, q = 1$.
- $\alpha = \epsilon_i + \epsilon_j - 2\epsilon_k,\ \beta = \epsilon_i - \epsilon_j \to \langle \beta, \alpha \rangle = 0$. $p = 0, q = 0$.
- $\alpha = \epsilon_i + \epsilon_j - 2\epsilon_k,\ \beta = \epsilon_i - \epsilon_k \to \langle \beta, \alpha \rangle = 1$. $p = 1, q = 0$.
- $\alpha = \epsilon_i + \epsilon_j - 2\epsilon_k,\ \beta = \epsilon_k - \epsilon_i \to \langle \beta, \alpha \rangle = -1$. $p = 0, q = 1$.