LECTURE 18: CLASSIFICATION OF ABSTRACT CARTAN MATRICES / DYNKIN DIAGRAMS

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1. Examples of Dynkin diagrams

In the examples and exercises that follow, we will compute the Cartan matrices (as described in the previous lecture) for the indecomposable root systems that we have encountered earlier. We record these as elegant Dynkin diagrams, summarized in Figure 1.1. Later in the lecture, we will prove that these are actually all the possible indecomposable root systems. We also compute extended Dynkin diagrams specifically for the purposes of this proof.

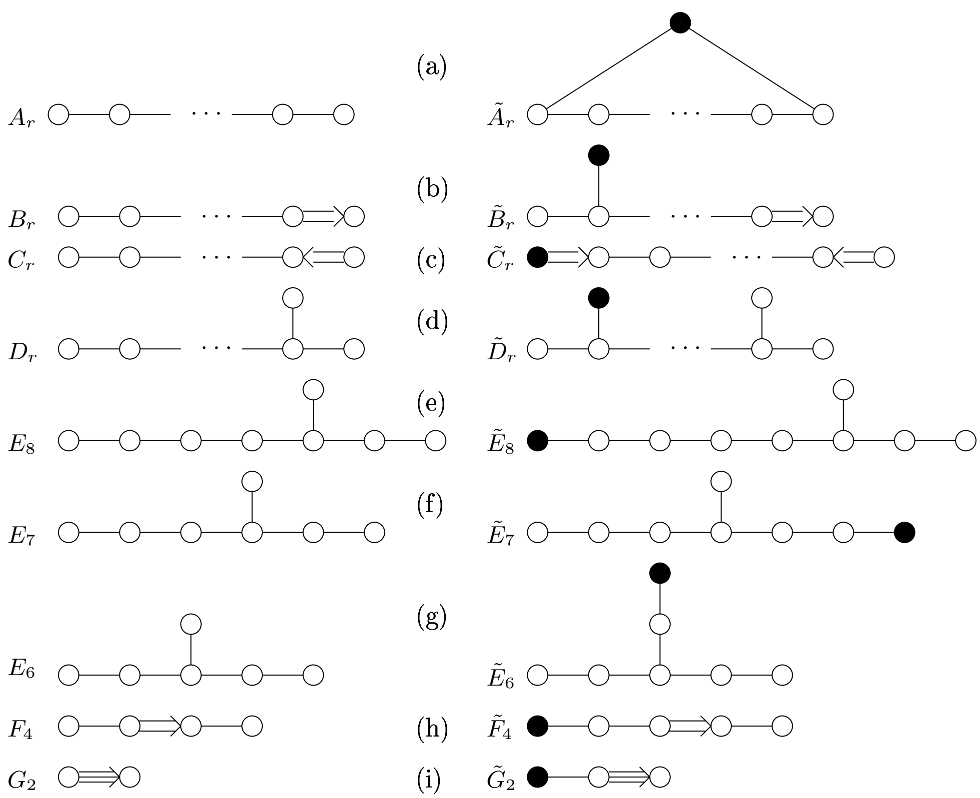

Figure 1.1. Dynkin diagrams of all the indecomposable root systems, from top to bottom: $A_r, B_r, C_r, D_r, E_8, E_7, E_6, F_4, G_2$.

In the following examples, the rank of the root system is always denoted by r, and we get simple roots $\alpha_1, \dots, \alpha_r$. The lowest negative root, used in the extended Dynkin diagram is denoted α_0. Perhaps the most detailed explanation of how to compute a Cartan matrix is given in Exercise 1.

Example 1.1. A_r: $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $\Delta = \Delta_{sl_{r+1}(\mathbb{F})} = \{\varepsilon_i - \varepsilon_j | i, j \in [r+1]\} \subset V$, where V is the subspace of $\bigoplus_{i=1}^{r+1} \mathbb{R}\varepsilon_i$, on which the sum of coordinates (in the basis $\{\varepsilon_i\}$) is zero.

Take $f \in V^*$ given by $f(\varepsilon_1) = r+1, f(\varepsilon_2) = r, \dots, f(\varepsilon_{r+1}) = 1$; hence, $f \neq 0$ on Δ, and f is integer-valued on all the roots.

Then, $\Delta_+ = \{\varepsilon_i - \varepsilon_j | i < j\}$. What is Π? Clearly, $\alpha = \varepsilon_i - \varepsilon_{i+1} \in \Pi$, since $f(\alpha) = 1$. There are r simple roots altogether (since Π spans V) and r such α, so $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}\}_{i=1}^r$.

For the Cartan matrix, recall that:

$$A = \left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \right)_{i,j=1}^r = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Hence, the Dynkin diagram in Figure 1.1a.

The largest root is $\theta = \varepsilon_1 - \varepsilon_{r+1}$, so $\alpha_0 = \varepsilon_{r+1} - \varepsilon_1$ (α_0 denotes the lowest negative root). This yields the extended Dynkin diagram (from the matrix on roots $\alpha_0, \alpha_1 \dots, \alpha_r$, denoted $\tilde{A} = \left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\right)$) in Figure 1.1a.

Example 1.2. B_r: $\Delta = \Delta_{so_{2r+1}(\mathbb{F})} = \{\pm\varepsilon_i \pm \varepsilon_j (i \neq j), \pm\varepsilon_i\}_{i,j=1}^r \subset V = \bigoplus_{i=1}^r \mathbb{R}\varepsilon_i$.

Take the f analogously to A_r (but we have r, rather than $r+1$ roots now): $f(\varepsilon_1) = r, \dots, f(\varepsilon_r) = 1$.

Then, $\Delta_+ = \{\varepsilon_i + \varepsilon_j (i \neq j), \varepsilon_i - \varepsilon_j (i < j), \varepsilon_i\}$, and $f = 1$ for $\varepsilon_i - \varepsilon_{i+1}$ and ε_r. Hence $\Pi = \{\varepsilon_i - \varepsilon_{i+1}\ (i = 1, \dots, r-1), \varepsilon_r\}$. The Cartan matrix is almost identical, except that $A_{r,r-1} = \frac{2(\alpha_r, \alpha_{r-1})}{(\alpha_r, \alpha_r)} = 2(\alpha_r, \alpha_{r-1}) = -2$, so

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -2 & 2 \end{pmatrix}.$$

The lowest root is $\alpha_0 = -(\varepsilon_1 + \varepsilon_2)$, so we get the Dynkin diagrams in Figure 1.1b.

Example 1.3. C_r: $\Delta = \Delta_{sp_{2r}(\mathbb{F})}$. The roots are the same as above, except $\pm\varepsilon_i$ becomes $\pm 2\varepsilon_i$; we also take the same f. Then, $f(\varepsilon_i - \varepsilon_{i+1}) = 1$ for $i = 1, 2, \dots, r-1$ (label these simple roots $\alpha_1, \dots, \alpha_{r-1}$), and we need an rth simple root. We might only possibly get $f = 2$ by summing some of the preceding α_i. However, it's impossible to obtain $2\varepsilon_r$ this way. Hence, $\Pi = \{\varepsilon_i - \varepsilon_{i+1}\ (i = 1, \dots, r-1),\ 2\varepsilon_r\}$. The new Cartan integers are $\frac{2(\alpha_{r-1}, \alpha_r)}{(\alpha_{r-1}, \alpha_{r-1})} = -2$, and $\frac{2(\alpha_r, \alpha_{r-1})}{(\alpha_r, \alpha_r)} = -1$, which gives us:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -2 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix},$$

and the Dynkin diagram in Figure 1.1c. The highest root is $\theta = 2\varepsilon_1$, so $\alpha_0 = -2\varepsilon_1$, and we get the extended Dynkin diagram in Figure 1.1c.

Example 1.4. D_r: $\Delta = \Delta_{so_{2r}(\mathbb{F})} = \{\pm\varepsilon_i \pm \varepsilon_j (i \neq j)\}_{i,j=1}^r$. Define $f \in V^*$ by $f(\varepsilon_1) = r-1, \ldots, f(\varepsilon_r) = 0$.

Then $\Delta = \{\varepsilon_i \pm \varepsilon_j (i < j)\}$, and $\Pi = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{r-1} - \varepsilon_r, \varepsilon_{r-1} + \varepsilon_r\}$ since we get $f(\alpha_i) = 1$ for all i.

Compute

$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & \ldots & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & 2 & -1 & -1 \\ \vdots & & 0 & -1 & 2 & 0 \\ 0 & \ldots & 0 & -1 & 0 & 2 \end{pmatrix},$$

so we have the Dynkin diagram in Figure 1.1d.

Then, $\theta = \varepsilon_1 + \varepsilon_2$, so $\alpha_0 = -\varepsilon_1 - \varepsilon_2$, so we get the $\tilde{D}_r$ in Figure 1.1d.

Example 1.5. E_8: $\Delta = \Delta_{E_8} = \{\pm\varepsilon_i \pm \varepsilon_j (i \neq j), \underbrace{\tfrac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \cdots \pm \varepsilon_8)}_{\text{(even number of + signs)}}\}_{i,j=1}^8$, with $V = \bigoplus_{i=1}^8 \mathbb{R}\varepsilon_i$. Let $f(\varepsilon_1) = 23, f(\varepsilon_2) = 6, f(\varepsilon_3) = 5, \ldots f(\varepsilon_8) = 0$; we have an even number of odd $f(\varepsilon_i)$, so f is integer on all roots. Then,

$$\Delta_+ = \{\varepsilon_i \pm \varepsilon_j (i < j), \frac{1}{2} \underbrace{(\varepsilon_1 \pm \varepsilon_2 \pm \cdots \pm \varepsilon_8)}_{\text{(even number of + signs)}} \}_{i,j=1}^8$$

and

$$\Pi = \{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \ldots \varepsilon_7 - \varepsilon_8, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \cdots - \varepsilon_7 + \varepsilon_8)\}.$$

In particular, $f(\alpha_i) = 1$ for all i; $\theta = \varepsilon_1 + \varepsilon_2$, so $\alpha_0 = -\varepsilon_1 - \varepsilon_2$ and we get the diagrams E_8 and $\tilde{E}_8$ in Figure 1.1e.

Exercise 1. *From Exercise 16.4, we have:*

$$\Delta_{E_7} = \{\varepsilon_i - \varepsilon_j (1 \leq i \neq j \leq 8), \frac{1}{2}(\underbrace{\pm\varepsilon_1 \pm \varepsilon_2 \pm \cdots \pm \varepsilon_8}_{4+\text{signs}})\}$$

$$\Delta_{E_6} = \{\varepsilon_i - \varepsilon_j (1 \leq i \neq j \leq 6), \pm(\varepsilon_7 - \varepsilon_8), \frac{1}{2}(\underbrace{\pm\varepsilon_1 \pm \varepsilon_2 \pm \cdots \pm \varepsilon_6}_{3+\text{signs}}) \pm \frac{1}{2}(\varepsilon_7 - \varepsilon_8))\}.$$

Perform the same analysis as in the Examples for E_7 and E_6. Show that their diagrams are as in Figure 1.1f,g.

For E_7, we will pick $f(\varepsilon_i)$ so that all roots have non-zero integer values. For this, we need $f(\varepsilon_i)$ to be distinct integers (hence $\pm\varepsilon_i \pm \varepsilon_j$ are all integer and non-zero). Additionally, we want an even number of the $f(\varepsilon_i)$ to be odd. That way, roots of the second type also have integer values. Take $f(\varepsilon_2) = 7, f(\varepsilon_3) = 6, \ldots, f(\varepsilon_8) = 1$; then, $\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \ldots, \alpha_6 = \varepsilon_7 - \varepsilon_8$ are all simple roots with value 1. We have $(\alpha_i, \alpha_j) = -1$ if $|i-j| = 1$ and 0 otherwise so far, and $(\alpha_i, \alpha_i) \equiv 2$. There is just one more simple root, and it must be of the second type. We need to choose $f(\varepsilon_1)$ appropriately. In order for f to be integer-valued on all roots, $f(\varepsilon_1)$ must be even. To avoid a zero-valued root, we will make it sufficiently large that

$$\frac{1}{2}(f(\varepsilon_1) - 7 - 6 - 5 - 4 + 3 + 2 + 1) > 0,$$

since that's the closest we can get to zero if $f(\varepsilon_1)$ is large. Simplifying, we have: $f(\varepsilon_1)$ even, and $f(\varepsilon_1) > 16$, so we'll take $f(\varepsilon_1) = 18$. Then $f(\alpha_7) = 1, \alpha_7 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8)$, and we have the last simple root. We have $(\alpha_i, \alpha_7) = 0$ for all $1 \leq i \neq 4 \leq 6$, with $(\alpha_4, \alpha_7) = -1$ and $(\alpha_7, \alpha_7) = 2$. Hence, all the connections are simple, and we have the right diagram. The highest root is clearly $\theta = \varepsilon_1 - \varepsilon_8 \overset{f}{\mapsto} 17$, so $\alpha_0 = \varepsilon_8 - \varepsilon_0$, which connects (simply) only with α_7, so we have the right extended diagram too.

For E_6, we choose the same $f(\varepsilon_2), \ldots f(\varepsilon_8)$. This gives us the following simple roots: $\alpha_1 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_4 = \varepsilon_5 - \varepsilon_6, \alpha_5 = \varepsilon_7 - \varepsilon_8$, with $(\alpha_i, \alpha_j) = -\delta_{ij}$ if $1 \leq i, j \leq 4$; all $(\alpha_i, \alpha_i) = 2$, and α_5 doesn't connect with any of these roots. Now, to choose the right $f(\varepsilon_1)$, we assume it's large and take the smallest non-negative number we can get:

$$\frac{1}{2}(f(\varepsilon_1) - 7 - 6 - 5 + 4 + 3 - 2 + 1 > 0,$$

so $f(\varepsilon_1) > 12$. As before, we want it to be even, and so pick $f(\varepsilon_1) = 14$, which gives the last simple root $f(\alpha_6) = 1, \alpha_6 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$. α_7 connects only with α_3 and α_5; $(\alpha_3, \alpha_6) = -1, (\alpha_5, \alpha_6) = -1$ and $(\alpha_6, \alpha_6) = 2$. So, the connections are all simple, and we get the correct diagram. The highest root in this case is $\theta = \varepsilon_1 - \varepsilon_6 \overset{f}{\mapsto} 11$, so $\alpha_0 = \varepsilon_6 - \varepsilon_1$, which connects only with α_4, which, again, gives the right extended diagram.

Exercise 2. *From Exercises 16.5 and 16.6, we have:*

$$\begin{aligned}\Delta_{F_4} &= \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i, \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}, 1 \leq i \neq j \leq 4 \\ \Delta_{G_2} &= \{\varepsilon_i - \varepsilon_j, \pm(\varepsilon_i + \varepsilon_j - 2\varepsilon_k)\}, 1 \leq i, j, k \leq 3, \textit{all distinct}\end{aligned}$$

Perform the same calculations for F_4 and G_2, and show that they have the diagrams in Figure 1.1h,i.

Here, we proceed exactly as in the previous exercise. For F_4, we pick $f(\varepsilon_1) = 8, f(\varepsilon_2) = 3, f(\varepsilon_3) = 2, f(\varepsilon_4) = 1$, which gives us the simple roots $\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4, \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$, with simple connections between α_1 and each of α_2, α_3, α_4. Additionally, there is a double-arrow connection from α_2 to α_3, so we get the correct diagram. $\alpha_0 = -\varepsilon_1 - \varepsilon_2 \overset{f}{\mapsto} -11$ connects up only with α_1, which gets us the extended diagram.

For G_2, pick $f(\varepsilon_1) = 4, f(\varepsilon_2) = 2, f(\varepsilon_3) = 1$, which gives us simple roots $\alpha_1 = \varepsilon_2 - \varepsilon_3$ and $\alpha_2 = \varepsilon_1 + \varepsilon_3 - 2\varepsilon_2$, with $(\alpha_1, \alpha_1) = 2$,$(\alpha_2, \alpha_2) = 6$, and $(\alpha_1, \alpha_2) = -3$,

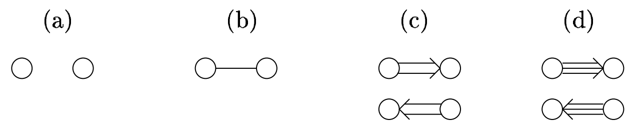

FIGURE 2.1. The four Dynkin diagram connection types, corresponding to the four types of 2×2 Cartan matrix minors.

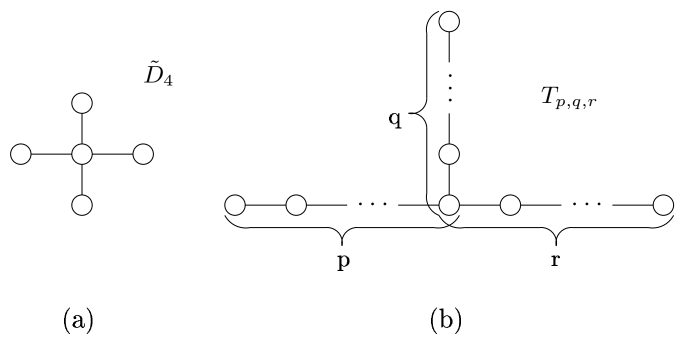

FIGURE 2.2. Some diagrams for the simply laced case.

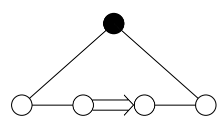

FIGURE 2.3. A loop with one double-connection.

which gives the desired connection. $\alpha_0 = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1 \overset{f}{\mapsto} -5$, with a simple connection to α_2, as in the extended diagram.

2. Classification of Dynkin Diagrams

Theorem 2.1. *The Dynkin diagrams of all indecomposable Cartan matrices are* $A_r, B_r, C_r, D_r, E_8, E_7, E_6, F_4, G_2$.

Proof. We have to choose connected graphs with connections of the 4 types depicted in Figure 2.1, such that the matrix of any subgraph has a positive determinant. In particular, our graphs contain no extended Dynkin diagrams as induced subgraphs, since these have determinant 0.

Part 1. Classify all "simply laced" Dynkin diagrams, i.e. using only 0- or 1-edge connections (which correspond to a symmetric A). Such a graph contains no cycles, since those are $\tilde{A}_r$. If there are no branching points, we get A_r. Next, such a graph contains at most 1 branching point, since otherwise it contains $\tilde{D}_r$. If there is a branching point, one has at most 3 branches, since $\tilde{D}_4$ is the 4-star in Figure 2.2a. Thus, we are left with a graph of the form $T_{p,q,r} (p \geq q \geq r \geq 2)$ depicted in Figure 2.2b. But the graph cannot contain any of $\tilde{E}_6 = T_{3,3,3}, \tilde{E}_7 = T_{4,4,2}, \tilde{E}_8 = T_{6,3,2}$, hence the only possibilities are D_r, E_6, E_7, E_8. Why? Suppose $r = 3$, then $p \geq q \geq 3$, so it contains $T_{3,3,3}$. Hence, $r = 2$. From $\tilde{E}_7$, we can't have $q \geq 4 \Rightarrow q = 2, 3$, and we get $p \leq 5$ likewise.

Part 2. Classify all non-simply laced diagrams, i.e. those containing double- or triple-edge connections (corresponding to non-symmetric A). This can be done by a more complicated case analysis. However, we'd like an effective way of ruling out diagrams like the one in Figure 2.3. This can be done by computing many large determinants, but we would rather argue using the following:

Exercise 3. *Prove the following two useful lemmas:*

Lemma 2.2. *Let*

$$A = \left(\begin{array}{c|cccc} 2 & -a & 0 & \cdots & 0 \\ \hline -b & & & & \\ 0 & & & & \\ \vdots & & A_{n-1} & & \\ 0 & & & & \end{array}\right).$$

Then, $\det A = 2\det A_{n-1} - ab\det A_{n-2}$*, where* A_{n-2} *is* A *with the first and second rows and columns deleted.*

Proof. Denote by $A\backslash(i,j)$ the matrix A with the ith row and jth column removed. In particular, $A_{n-2} = A\backslash(1,2)\backslash(1,1)$.

To prove this lemma, we expand the determinant by minors along the first row, getting $\det A = 2\det A_{n-1} + a\det A\backslash(1,2)$; then, we expand $\det A\backslash(1,2)$ along its first column, which gives $\det A = 2\det A_{n-1} - ab\det A_{n-2}$, as desired. □

Lemma 2.3. *Let*

$$A = \begin{pmatrix} c_1 & -a_1 & 0 & \cdots & 0 & -b_n \\ -b_1 & c_2 & -a_2 & 0 & & 0 \\ 0 & -b_2 & c_3 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & -a_{n-2} & 0 \\ 0 & & \ddots & -b_{n-2} & c_{n-1} & -a_{n-1} \\ -a_n & 0 & \cdots & 0 & -b_{n-1} & c_n \end{pmatrix}.$$

Then, $\det(A - \epsilon E_{12}) = \det A - \epsilon(b_1\det A_{n-2} + a_2a_3\ldots a_n)$*. In particular, if* $\det A_{n-2} > 0$*,* $b_1 > 0$*,* $a_2\ldots a_n > 0$*, and* $\epsilon > 0$*, then* $\det A - \epsilon E_{12} < \det A$*.*

Proof. To prove this lemma, we expand the determinant of $A - \epsilon E_{12}$ along the second column to get

$$(a_1+\epsilon)\det A\backslash(1,2) + c_2\det A\backslash(2,2) + b_2\det A\backslash(3,2) = \det A + \epsilon\det A\backslash(1,2).$$

$A\backslash(1,2)$ looks like this:

$$\begin{pmatrix} -b_1 & -a_2 & 0 & \cdots & 0 \\ 0 & c_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -a_{n-2} & 0 \\ 0 & \ddots & -b_{n-2} & c_{n-1} & -a_{n-1} \\ -a_n & \cdots & 0 & -b_{n-1} & c_n \end{pmatrix}.$$

If we expand its determinant along the first column, we get $-b_1\det A\backslash(1,2)\backslash(1,1) + (-1)^{n-1}a_n\det A\backslash(1,2)\backslash(n-1,1)$. The first part of the expression is simply $-b_1\det A_{n-2}$.

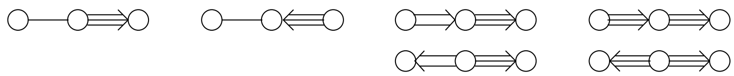

FIGURE 2.4. Possible neighbors of G_2 in a diagram.

As for the second part, the matrix we get is:

$$\begin{pmatrix} -a_2 & 0 & \cdots & 0 \\ c_3 & \ddots & \ddots & \vdots \\ \ddots & \ddots & -a_{n-2} & 0 \\ \ddots & -b_{n-2} & c_{n-1} & -a_{n-1} \end{pmatrix},$$

which is lower triangular with a diagonal consisting of all a_i, $2 \leq i \leq n-1$, hence with determinant $(-1)^{n-2}a_2a_3\ldots a_{n-1}$. Putting all this together, we get the desired

$$\det A + \epsilon(-b_1 \det A_{n-2} - a_2a_3\ldots a_n).$$

□

Lemma 2.3 implies that in the non-simply laced case, there are no cycles as well. This is because $\det \tilde{A}_{r-1} = 0$; so, any other cycle with subdiagrams of positive determinant must have determinant < 0 by the lemma. For example, the diagram in Figure 2.3 has $A = \tilde{A}_4 - E_{12}$, so $\det A < 0$.

Next, looking at the extended Dynkin diagrams we calculate that if the diagram contains G_2 (Figure 1.1i) it must be G_2, by Lemma 2.2. Otherwise, it must contain one of the possibilities in Figure 2.4, none of which have suitable determinant: their Cartan matrices are of the form

$$A = \begin{pmatrix} 2 & -a & 0 \\ -b & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix} \Rightarrow A_{n-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \text{ and } A_{n-2} = \begin{pmatrix} 2 \end{pmatrix}.$$

Hence, by Lemma 2.2, $\det(A) = 2\det(A_{n-1}) - ab\det(A_{n-2}) = 2(1-ab)$, which is zero or negative since $a, b > 0$. (The matrix of the second graph in the figure is actually A^T, but all the determinants are the same.)

It remains to look at the case when we have only simple or double connections. Looking at the extended Dynkin diagrams, $\tilde{C}_r$ (in Figure 1.1c) cannot be a subdiagram. The variants with flipped arrow directions also don't work. They are obtained from the extended Cartan matrix $\tilde{A}$ of $\tilde{C}_r$ by replacing some of $\tilde{A}, \tilde{A}_{n-1}, \tilde{A}_{n-2}$ by their transposes, which doesn't change any of the determinants in the calculation. Thus, by Lemma 2.2, their determinants are also 0.

Therefore only one double connection is possible. But, then we cannot have branching points, since $\tilde{B}_r$ contains a double edge and a branching point. So, the only remaining case is a line with a left-right double edge, having p single edges to the left, and q single edges to the right. $\tilde{F}_4$ has $p = 2, q = 1$; its transpose has $p = 1, q = 2$. Hence, if the diagram contains $\tilde{F}_4$, it must be $\tilde{F}_4$. Otherwise, either $p = 0$ or $q = 0$, and we have B_r or C_r.

□

We have now shown that any finite-dimensional simple Lie algebra yields one of a very restricted set of Dynkin diagrams (and hence Cartan matrices). The next step in the classification of semisimple Lie algebras will be to give a construction associating an abstract Cartan matrix to a Lie algebra, and hence to prove that these four classes plus five exceptional algebras are in fact the only finite-dimensional simple Lie algebras.