18.745 Introduction to Lie Algebras Fall 2004

Lecture 19 — November 16, 2004

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We have been proving the following,

Theorem 1 *(Classification theorem) Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0. Then $\mathfrak{g}$ is isomorphic to a direct sum of simple Lie algebras, and a complete and non-redundant list of the latter is as follows:*

$$\mathfrak{sl}_n(\mathbb{F})\ (n \geq 2), \mathfrak{so}_n(\mathbb{F})\ (n \geq 7), \mathfrak{sp}_n(\mathbb{F})\ (n \geq 4, \textit{even}), E_6, E_7, E_8, F_4, G_2.$$

The strategy of the proof is given in the following diagram.

$$\begin{array}{ccccccccc} \mathfrak{g} & \xrightarrow{\text{choose } \mathfrak{h}} & \underset{\text{root system}}{\Delta} & = & \text{abstract root system} & \xrightarrow{\text{choose } f \in V^*} & \Pi \longrightarrow & \underset{\text{Cartan matrix}}{A} & = \text{abstract Cartan matrix } A \\ \uparrow & & & & & & & & \\ \mathfrak{g}(A) & \longleftarrow & & & & & & & \end{array}$$

First construct the following set of generators E_i, F_i, H_i $(i = 1, \ldots, r = \dim \mathfrak{h} = \text{rank}\ (\mathfrak{g}))$ as follows: let α_i be a simple root, and choose $E_i \in \mathfrak{g}_{\alpha_i}, F_i \in \mathfrak{g}_{-\alpha_i}$ such that $[E_i, F_i] = \frac{2\nu^{-1}(\alpha_i)}{K(\alpha_i, \alpha_j)}$ (recall that $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}] = \mathbb{F}\nu^{-1}(\alpha_i), \mathfrak{g}_{\alpha_i} = \mathbb{F}E_i, \mathfrak{g}_{-\alpha_i} = \mathbb{F}F_i$.) Then we have:

$$[H_i, H_j] = 0, [H_i, E_j] = a_{ij}E_j, [H_i, F_j] = -a_{ij}F_j, [E_i, F_j] = \delta_{ij}H_i \qquad (*)$$

where $a_{ij} := \frac{2K(\alpha_i, \alpha_j)}{K(\alpha_i, \alpha_i)}$ are the entries of the Cartan matrix $A = (a_{ij})$.

To check these: $[H_i, E_j] = \alpha_j(H_i)E_j = \frac{2\alpha_j(\nu^{-1}(\alpha_i))}{K(\alpha_i, \alpha_i)} = \frac{2K(\alpha_j, \alpha_i)}{K(\alpha_i, \alpha_i)} = a_{ij}$; $[H_i, F_j] = -a_{ij}F_j$ is clear; $[E_i, F_i]$ has been checked. $[E_i, F_j] = 0$ because it belongs to $\mathfrak{g}_{\alpha_i - \alpha_j}$, so it is not a root by part (a) of the Theorem in Lecture 17.

Next, we denote by $\mathfrak{n}_+$ (respectively $\mathfrak{n}_-$) the subalgebra of $\mathfrak{g}$ generated by all E_is (respectively F_is). Then

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha, \quad \mathfrak{n}_- = \bigoplus_{\alpha \in -\Delta_+} \mathfrak{g}_\alpha.$$

Indeed, let $\alpha \in \Delta_+ \setminus \Pi$. Then by part (c) of the Theorem from Lecture 17, $\alpha - \alpha_i \in \Delta_+$ for some simple root α_i, hence $\mathfrak{g}_\alpha = [\mathfrak{g}_{\alpha - \alpha_i}, E_i]$ since $\dim \mathfrak{g}_\alpha = 1$. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ if $\alpha, \beta, \alpha + \beta \in \Delta$ by part (d) of Theorem 3 from Lecture 13.

This proves that all $\mathfrak{g}_\alpha$, with $\alpha \in \Delta_+$, are in $\mathfrak{n}_+$. Likewise all $\mathfrak{g}_{-\alpha} (\alpha \in \Delta_+)$ are in $\mathfrak{n}_-$. Note that we have the so-called triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ (a direct sum as vector spaces).

Exercise 19.1. (a) Show that for $\mathfrak{sl}_n$, $\mathfrak{n}_\pm$ = strictly $\substack{\text{upper} \\ \text{lower}}$ triangular matrices, if $\mathfrak{h}$ = diagonal matrices, and if f chosen as we did. (b) Find the triangular decompositions for $\mathfrak{so}_n$ and $\mathfrak{sp}_n$.

Solution. (a) In Lecture 14, we noted that

$$\mathfrak{sl}_n(\mathbb{F}) = \mathfrak{h} \oplus \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbb{F}E_{ij} \right),$$

where $\mathfrak{h}$ was the diagonal matrices, and $\mathbb{F}E_{ij}$ is the root space attached to the root $\varepsilon_i - \varepsilon_j$. We need to distinguish the positive roots from the negative roots; the f we have chosen gives positive roots for each E_{ij} where $i > j$. The direct sum of the spaces $\mathbb{F}E_{ij}$ where $i > j$ is indeed the desired space $\mathfrak{n}_+$ of strictly upper triangular matrices, and $\mathfrak{n}_-$ is the space of strictly lower triangular matrices, showing the triangular decomposition.

(b) For $\mathfrak{so}_n(\mathbb{F})$, we have

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & & & & 0 \\ & a_2 & & & \\ & & \ddots & & \\ & & & -a_2 & \\ 0 & & & & -a_1 \end{pmatrix} \right\}.$$

and $\mathfrak{so}_n(\mathbb{F}) = \mathfrak{h} \oplus \left(\bigoplus_{i,j} \mathbb{F}F_{ij} \right)$, where $F_{ij} = E_{ij} - E_{n+1-j,n+1-i}$. Then the positive roots are $\varepsilon_i \pm \varepsilon_j (i < j), \varepsilon_i$. Hence the space $\mathfrak{n}_+$ is the set of matrices A such that $A' = -A$ which are strictly upper-triangular, while the $\mathfrak{n}_-$ is the set of matrices A such that $A' = -A$ which are strictly lower-triangular.

For $\mathfrak{sp}_n(\mathbb{F})$, we have the same Cartan subalgebra $\mathfrak{h}$; this time, the $F_{ij} = E_{ij} - E_{n+1-j,n+1-i}$ for $1 \leq i, j \leq r (i \neq j)$ and $F_{ij} = E_{ij} + E_{n+1-j,n+1-i}$ $(1 \leq i \leq r, r+1 \leq j \leq n$. The corresponding root vectors are $\Delta_{\mathfrak{sp}_{2r}} = \{\varepsilon_i - \varepsilon_j (i, j = 1, \ldots, r, i \neq j), \varepsilon_i + \varepsilon_j, -\varepsilon_i - \varepsilon_j (i, j = 1, \ldots, r)\}$; the positive roots are now $\varepsilon_i \pm \varepsilon_j (i < j)$ and $2\varepsilon_i$ if n is even. Finally, $\mathfrak{n}_+$ is the set of upper triangular matrices (in block form) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $d = -a', b = b', c = c'$, where $'$ denotes transposition with respect to the antidiagonal, and $\mathfrak{n}_-$ is the set of lower triangular matrices which satisfy this condition.

Remark 1 *Any non-zero ideal of $\mathfrak{g}$ has a non-zero intersection with $\mathfrak{h}$.*

Lemma 1 *Let $\mathfrak{h}$ be a finite-dimensional abelian Lie algebra, and let π be a diagonalizable representation of $\mathfrak{h}$ in a vector space V (not necessarily finite-dimensional), i.e. $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\alpha$, where $V_\lambda = \{v \in V \mid \pi(h)v = \lambda(h)v, h \in \mathfrak{h}\}$. Then for any $\pi(\mathfrak{h})$-invariant subspace $U \subset V$, we have $U = \bigoplus_{\lambda \in \mathfrak{h}^*} (U \cap V_\lambda)$.*

Proof. Take $u \in U$ and write $u = \sum_\lambda v_\lambda$, where $v_\lambda \in V_\lambda$. Take $h \in \mathfrak{h}$ such that $\lambda_1(h) \neq \lambda_2(h)$ (scale so that $\lambda_1(h) = 1$.) Then $U \ni \pi(h) = \lambda_1(h)v_{\lambda_1} + \lambda_2(h)v_{\lambda_2} + \cdots$. Hence

$$U - \pi(h)u = (1 - \lambda_2(h))v_{\lambda_2} + (1 - \lambda_3(h))v_{\lambda_3} + \cdots,$$

and we may apply induction on the number of summands in u. $\square$

Proof of the Remark. If I is a non-zero ideal of $\mathfrak{g}$ which intersects $\mathfrak{h}$ trivially, then by the lemma, $\mathfrak{g}_\alpha \subset I$ for some root α. But then $[\mathfrak{g}_\alpha, g_{-\alpha}] = \mathbb{F}\nu^{-1}(\alpha) \subset I$, a contradiction. □

Let $\tilde{\mathfrak{g}}(A)$ be the Lie algebra on generators E_i, F_i, H_i $(i = 1, \ldots, r)$ subject to relations (*).

Exercise 19.2. Show that $\tilde{\mathfrak{g}}((2)) = \mathfrak{sl}_2(\mathbb{F})$ but $\tilde{\mathfrak{g}}\left(\left(\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array}\right)\right)$ is infinite dimensional. Find the elements of $\mathcal{J}\left(\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array}\right)$.

Solution. For the case $\tilde{\mathfrak{g}}((2)) = \mathfrak{sl}_2(\mathbb{F})$, we have $r = 1$, giving the generators E_1, F_1, H_1 with the relationships $[E_1, F_1] = H_1; [H_1, H_1] = 0; [H_1, E_1] = 2E_1; [H_1, F_1] = -2F_1$. Let

$$H_1 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right); E_1 = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right); F_1 = \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right);$$

simple calculations show that these relationships are satisfied, and these matrices generate $\mathfrak{sl}_2(\mathbb{F})$.

To show that $\tilde{\mathfrak{g}}\left(\left(\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array}\right)\right)$ is infinite-dimensional, we note that $\tilde{\mathfrak{n}}_+$ and $\tilde{\mathfrak{n}}_-$ are freely generated over the generators E_1, E_2, which is an infinite-dimensional space. Since $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_+ \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_-$, it is also infinite-dimensional.

The maximal ideal $\mathcal{J}\left(\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array}\right)$ is the ideal containing the inverse images of the triple commutators $E_1E_2E_1$ and $E_2E_1E_2$, as well as $F_1F_2F_1$ and $F_2F_1F_2$. □

Lemma 2 *(a)* $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$, *where* $\tilde{\mathfrak{n}}_+$ *(resp.* $\tilde{\mathfrak{n}}_-$*) is generated by the* E_i*s (resp.* F_i*s) and* $\mathfrak{h}$ = *span of* H_i*s. (b)* $\tilde{\mathfrak{g}}(A)$ *has a maximal ideal* $\mathcal{J}(A)$ *among ideals intersecting* $\mathfrak{h}$ *trivially.*

Proof. (**Exercise 19.3.**) First we show by induction on n that any commutator of length n:

$$[[b_{i_1}, b_{i_2}], \ldots, b_{i_n}] \quad (\text{where } b_{i_k} \in \{E_i, F_i, H_i\}),$$

lie in either $\mathfrak{n}_+$ or in $\mathfrak{n}_-$ or in $\mathfrak{h}$.

Solution. The base case is obvious. Now assume $b = [[b_{i_1}, b_{i_2}], \ldots, b_{i_n}]$ is in $\mathfrak{n}_+$, $\mathfrak{n}_-$ or $\mathfrak{h}$; we show that $b' = [[[b_{i_1}, b_{i_2}], \ldots, b_{i_n}], b_{i_{n+1}}]$ is in $\mathfrak{n}_+$, $\mathfrak{n}_-$ or $\mathfrak{h}$. The following table summarizes the possible values of the $n+1$-length commutator b', ignoring irrelevant constant factors in front of the b.

b \ $b_{i_{n+1}}$	E_i	F_i	H_i
E_j	0	$\delta_{ji}H_j$	$-a_{ij}E_j$
F_j	$-\delta_{ij}H_i$	0	$-a_{ij}F_j$
H_j	$a_{ji}E_i$	$-a_{ji}F_i$	0

Recalling that $\mathfrak{n}_+$ is defined to be the subalgebra generated by the E_i, that $\mathfrak{n}_-$ is generated by the F_i and that $\mathfrak{h}$ is generated by the H_i, we observe that b', the $n+1$-length commutator, always belongs to one of $\mathfrak{n}_+, \mathfrak{n}_-$ or $\mathfrak{h}$ if all the b_{i_k}s are E_i, F_i or H_i. $\square$

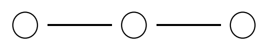
Hence $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}_+}$. This sum is direct since $\tilde{\mathfrak{n}}_+ = \bigoplus \tilde{\mathfrak{g}}(A)_\alpha, \alpha = \sum_{i=1}^n k_i \alpha_{ij}, k_i \in \mathbb{Z}_i$, not all 0, and similarly for $\tilde{\mathfrak{n}}_-$, but $\mathfrak{h} = \tilde{\mathfrak{g}}(A)_0$. (This proves (a).)

But by Lemma 1, any ideal I of $\tilde{\mathfrak{g}}(A)$ is of the form $I = I_- \oplus I_0 \oplus I_+$, where $I_\pm \subset \tilde{\mathfrak{n}}_\pm$, $I_0 \subset \mathfrak{h}$. So if $I \cap \mathfrak{h} = 0$, then $I = I_- \oplus I_+, I_\pm \subset \tilde{\mathfrak{n}}_\pm$. Hence (b) follows, since $\mathcal{J}(A)$, as a sum of all ideals intersecting $\mathfrak{h}$ trivially, is a proper maximal ideal. We let $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\mathcal{J}(A)$. Now consider the surjective homomorphism $\tilde{\mathfrak{g}}(A) \xrightarrow{\varphi} \mathfrak{g}$ defined by $\varphi(E_i) = E_i, \varphi(F_i) = F_i, \varphi(H_i) = H_i$ (which generate $\mathfrak{g}$). Moreover, $\mathcal{J}(A) \subset \ker \varphi$, otherwise $\varphi(\mathcal{J}(A))$ contradicts Remark 1, and furthermore, $\mathcal{J}(A) = \ker\varphi$ since $\mathcal{J}(A)$ is maximal. Hence we have an isomorphism $\mathfrak{g}(A) \tilde{\rightarrow} \mathfrak{g}$. (Note that φ is mapping the infinite-dimensional $\mathfrak{n}_+$ and $\mathfrak{n}_-$ to finite-dimensional $\tilde{\mathfrak{n}}_+$ and $\tilde{\mathfrak{n}}_-$). $\square$

So, if we know that $\mathfrak{g}$ with a given root system exists, like $\mathfrak{g} = \mathfrak{sl}_n, \mathfrak{so}_n$, or $\mathfrak{sp}_n$, then $\mathfrak{g}(A) = \mathfrak{g}$ is the Lie algebra with this root system. We have proved that any simple Lie algebra over $\mathbb{F}$ is isomorphic to $\mathfrak{sl}_n(\mathbb{F}), \mathfrak{so}_n(\mathbb{F}), \mathfrak{sp}_n(\mathbb{F})$, or possibly the simple Lie algebras whose root systems are $\Delta_{E_i}, i = 6, 7, 8, \Delta_{F_4}, \Delta_{G_2}$, called the exceptional Lie algebras. So it remains to prove the existence of the latter. Note that we have the same Cartan matrices in the following cases:

$A_1 = B_1 = C_1$ (1-d root systems are identical)

$B_2 = C_2$ 

$D_3 = A_3$ 

$D_2 = A_1 \oplus A_1$

or, on the level of Lie algebras, $\mathfrak{sl}_2(\mathbb{F}) \simeq \mathfrak{so}_3(\mathbb{F}) \simeq \mathfrak{sp}_2(\mathbb{F}); \mathfrak{so}_5(\mathbb{F}) \simeq \mathfrak{sp}_4(\mathbb{F}); \mathfrak{so}_6(\mathbb{F}) \simeq \mathfrak{sl}_4(\mathbb{F}); \mathfrak{so}_4 \simeq \mathfrak{sl}_2(\mathbb{F}) \oplus \mathfrak{sl}_2(\mathbb{F})$.

Now we want to construct the simple Lie algebras from their root systems. (We will carry out an explicit construction; this construction doesn't show uniqueness.)

First we consider the simply-laced case $A = A^T$, i.e. all roots have the same length. Let (V, Δ) be a simply-laced root system, with $(\cdot,\cdot)$ such that $(\alpha, \alpha) = 2$ for any root α, and let $Q = \mathbb{Z}\Delta$ be the root lattice. Then $\Delta = \{\alpha \in Q \mid (\alpha, \alpha) = 2\}$, in all examples A_r, D_r, E_6, E_7, E_8; the first two are known to exist, whereas the last three are new.

Consider the following space over $\mathbb{F}$: $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta} \mathbb{F}E_\alpha)$, where $\mathfrak{h} = \mathbb{F} \otimes_{\mathbb{Q}} (\mathbb{Q}\Delta)$ and E_α are some non-zero vectors. Extend the bilinear form from $(\cdot,\cdot)$ on V to $\mathfrak{h}$ by bilinearity. Define brackets on $\mathfrak{g}$ as follows:

(i) $[h, h'] = 0$ for $h, h' \in \mathfrak{h}$.

(ii) $[h, E_\alpha] = (h, \alpha)E_\alpha$

(iii) $[E_\alpha, E_{-\alpha}] = -\alpha$ (for convenience)

(iv) $[E_\alpha, E_\beta] = 0$ if $\alpha + \beta \notin \Delta \cup \{0\}$

(v) $[E_\alpha, E_\beta] = \varepsilon(\alpha, \beta)E_{\alpha+\beta}$ if $\alpha + \beta \in \Delta$.

Next time, we will study how to construct ε so that the Jacobi identity holds. To do this, we have to check the Jacobi identities of any triple of distinct basis elements of $\mathfrak{g}$:

(i) If this triple is $h, h', h'' \in \mathfrak{h}$, the identity is obvious.

(ii) If h, h', E_α or h, E_α, E_β, the Jacobi identity holds for any choice of $\varepsilon(\alpha, \beta)$ (**Exercise 19.4**).

(iii) (next time) If $E_\alpha, E_\beta, E_\gamma$, we will have to choose ε appropriately.

Solution. A straightforward calculation shows the Jacobi identity:

$$\begin{aligned}
[h,[h',E_\alpha]] + [h',[E_\alpha,h]] + [E_\alpha,[h,h']] &= [h,[h',E_\alpha]] + [h',[E_\alpha,h]] \\
&= [h,(h',\alpha)E_\alpha] + [h',-(h,\alpha)E_\alpha] \\
&= (h',\alpha)[h,E_\alpha] - (h,\alpha)[h',E_\alpha] \\
&= (h',\alpha)(h,\alpha)E_\alpha - (h,\alpha)(h',\alpha)E_\alpha \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
& [h,[E_\alpha,E_\beta]] + [E_\alpha,[E_\beta,h]] + [E_\beta,[h,E_\alpha]] \\
= \; & (\alpha+\beta,h)[E_\alpha,E_\beta] - (\beta,h)[E_\alpha,E_\beta] + (\alpha,h)[E_\beta,E_\alpha] \\
= \; & (\alpha+\beta,h)[E_\alpha,E_\beta] - (\beta,h)[E_\alpha,E_\beta] - (\alpha,h)[E_\alpha,E_\beta] \\
= \; & 0.
\end{aligned}$$

□