18.745 Lecture Notes - Lecture 2
Introduction and Basic Definitions - Part II

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A *homomorphism* of Lie algebras $\mathfrak{g}$ is a linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{g}_1$ which preserves the bracket, that is, $\phi([a,b]) = [\phi(a), \phi(b)]$. The *kernel* of a homomorphism ϕ, denoted $\operatorname{Ker}\phi$, is a subset of $\mathfrak{g}$ such that $\operatorname{Ker}\phi = \{a \in \mathfrak{g} | \phi(a) = 0\}$. The *image* of a such a homomorphism, $\operatorname{Im}\phi \equiv \{\phi(a) | a \in \mathfrak{g}_1\}$ is a subset of $\mathfrak{g}_1$. A homomorphism ϕ is called an *isomorphism* if it is bijective, i.e. if $\operatorname{Ker}\phi = 0$, $\operatorname{Im}\phi = \mathfrak{g}_1$.

Exercise 2.1: Prove the following claims:

a) $\operatorname{Ker}\phi$ is an ideal in $\mathfrak{g}$.

b) $\operatorname{Im}\phi$ is a subalgebra of $\mathfrak{g}$.

c) The Lie algebra $\operatorname{Im}\phi$ is isomorphic to the Lie algebra $\mathfrak{g}/\operatorname{Ker}\phi$.

Proof.

a) Given $a \in \operatorname{Ker}\phi, b \in \mathfrak{g}$. $\phi([a,b]) = [\phi(a), \phi(b)] = [0, \phi(b)] = 0 \rightarrow [a,b] \in \operatorname{Ker}\phi \rightarrow [\operatorname{Ker}\phi, \mathfrak{g}] \subset \operatorname{Ker}\phi \rightarrow \operatorname{Ker}\phi$ is an ideal.

b) $a, b \in \operatorname{Im}\phi \rightarrow a = \phi(a'), b = \phi(b') \rightarrow \phi(\lambda a' + \nu b') = \lambda a + \nu b$ (by the linearity of ϕ) $\rightarrow \lambda a + \nu b \in \operatorname{Im}\phi \rightarrow \operatorname{Im}\phi$ a subspace.
$[a,b] = [\phi(a'), \phi(b')] = \phi([a', b']) \rightarrow [a,b] \in \operatorname{Im}\phi \rightarrow \operatorname{Im}\phi$ a subalgebra of $\mathfrak{g}$

c) Define $\psi : \mathfrak{g}/\operatorname{Ker}\phi \rightarrow \operatorname{Im}\phi, \psi(a + \operatorname{Ker}\phi) = \phi(a)$. $\psi(\mathfrak{g}/\operatorname{Ker}\phi)$ is evidently equal to $\operatorname{Im}\phi$, which implies ψ is surjective.
ψ is well defined, since choosing any representatives $c, d \in a + \operatorname{Ker}\phi$ will both yield $\phi(a)$ under ψ.
$\psi(a + \operatorname{Ker}\phi) = \psi(b + \operatorname{Ker}\phi) \rightarrow \phi(a) = \phi(b) \rightarrow a = b + c, c \in \operatorname{Ker}\phi \rightarrow a + \operatorname{Ker}\phi = b + \operatorname{Ker}\phi \rightarrow \psi$ is bijective, which implies that ψ is an isomorphism.

□

The *center* of a Lie algebra $\mathfrak{g}$, denoted $\mathrm{Z}(\mathfrak{g})$, is the set of elements commuting with $\mathfrak{g}$ i.e. $\mathrm{Z}(\mathfrak{g}) = \{a \in \mathfrak{g} | [a, \mathfrak{g}] = 0\}$. This is obviously an ideal.

Exercise 2.2: If $\mathfrak{g}$ is a non-abelian Lie algebra, then $\dim \mathrm{Z}(\mathfrak{g}) \leq \dim(\mathfrak{g}) - 2$.

Proof. $\mathfrak{g}$ non-abelian implies that there are at least two linearly independent elements $a, b \in \mathfrak{g}$ such that $[a,b] = c, c \neq 0 \rightarrow a, b \notin \mathrm{Z}(\mathfrak{g}) \rightarrow \dim \mathrm{Z}(\mathfrak{g}) \leq \dim(\mathfrak{g}) - 2$ since the subspace spanned by a and b is not in $\mathrm{Z}(\mathfrak{g})$: $[Aa + Bb, a] = -Bc$, and $[Aa + Bb, b] = Ac$ for $A, B \in \mathbb{F}$, which are only 0 for B and A equal to 0 respectively. □

Exercise 2.3: If $\mathfrak{g}$ is a Lie algebra of dimension $n \geq 3$ with a 1 dimensional derived subalgebra, then $\mathfrak{g}$, up to isomorphism, is one of the following:

- $b_2 \oplus ab_{n-2}$
- $H_k \oplus ab_{n-2k-1}$

where b_2 is the unique two-dimensional non-abelian Lie algebra, ab_k is a k-dimensional abelian Lie algebra, and H_k is the k-th Heisenberg algebra of dimension $2k+1$.

Proof. $[\mathfrak{g}, \mathfrak{g}]$ 1 dimensional implies that $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}a, a \in \mathfrak{g}, a \neq 0$.

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Case 1 $\exists b \in \mathfrak{g} | [b,a] \neq 0 \rightarrow [b,a] = \lambda a$ ($\lambda \neq 0$). Send $b \rightarrow \frac{1}{\lambda}b$, that is, instead choose $b' = \frac{1}{\lambda}b$ and take that as a basis element. Then $[b,a] = a$. We can now construct a basis for the remaining $n-2 \geq 1$ dimensions of the algebra: Take any element that is linearly independent from a and b, call it e_1. $[a,e_1] = \lambda_1 a \rightarrow [a, e_1 + \lambda_1 b] = 0$, so send $e_1 \rightarrow e_1 + \lambda_1 b$. $[b,e_1] = \lambda_2 a \rightarrow [b, e_1 - \lambda_2 a] = 0$, and also $[a, e_1 - \lambda_2 a] = 0 \rightarrow$ sending $e_1 \rightarrow e_1 - \lambda_2 a$ creates e_1 that commutes with both a and b, while leaving e_1 linearly independent of a and b. Now we can construct a mutually commuting basis $\{e_1, \ldots, e_{n-2}\}$ that commutes with a and b by induction: Given $\{e_1, \ldots, e_i\}$ mutually commuting and commuting with a and b, select e_{i+1} linearly independent of $\{a, b, e_1, \ldots, e_i\}$, which can be done if $i \leq n+2$. e_{i+1} can be made to commute with a and b by the same process as before. Furthermore, given any $e_a, [e_{i+1}, e_a] = \lambda a$, and thus by the Jacobi identity $\lambda a = [b, [e_{i+1}, e_a]] = [[b, e_{i+1}], e_a] + [e_{i+1}, [b, e_a]] = 0$ and thus e_{i+1} will commute with all the e_a for $a \leq i+1$. Thus we have $\{a, b\}$ composing b_2, the unique two dimensional non-abelian Lie algebra, and $\{e_1, \cdots, e_{n-2}\}$ which span an $n-2$ dimensional abelian Lie algebra, and thus $\mathfrak{g} = b_2 \oplus ab_{n-2}$

Case 2 $\nexists b \in \mathfrak{g} | [b,a] \neq 0, \mathfrak{g}$ non-abelian since $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and thus $\exists p_1, q_1 \in \mathfrak{g} | [p_1, q_1] \neq 0 \rightarrow [p_1, q_1] = \lambda a$, sending $p_1 \rightarrow \frac{1}{\lambda} p_1 \rightarrow [p_1, q_1] = a$. Now if $\mathfrak{g}$ is three dimensional we have $\mathfrak{g} = H_1$, otherwise we can construct a basis for the remaining $n-3 \geq 1$ dimensional subspace. Choose d linearly independent from p_1, q_1, and a. $[d,a] = 0, [d, p_1] = \lambda_1 a \rightarrow$ send d to $d + \lambda_1 q_1 \rightarrow [d, p_1] = 0, [d, q_1] = \lambda_2 a \rightarrow$ send d to $d - \lambda_2 p_1 \rightarrow [d, q_1] = 0$ while keeping $[d, p_1] = 0$. Thus d commutes with the vectors p_1, q_1, a. While we do not have enough basis vectors we can use the previous algorithm to construct more. Now that we have $\{v_1, \ldots, v_{n-3}\}$ such that these all commute with $\{a, p_1, q_1\}$. Let $H = \{p_1, q_1\}$. We can put the basis into an appropriate form as follows:

1. Consider the complement subspace H^c to H. Since a is in this subspace, the subspace is a Lie algebra.
2. If $[H^c, H^c] = 0$, all the vectors in H^c commute and we are done.
3. If $[H^c, H^c] \neq 0, \exists p, q \in H^c | [p,q] = \lambda a \neq 0$, sending $p \rightarrow \frac{1}{\lambda} p$ will make $[p,q] = a$. We can also make p, q commute with all the vectors in H by taking the vectors in H pairwise and performing the same process as before. Move the new vectors we have created from H^c to H and go to the first step (recalculate H^c).

Since the last step decreases the dimension of H^c and $\dim(H^c)$ begins at a finite number, the algorithm must terminate, and then $H \cup \{a\}$ forms the Heisenberg algebra $H_{\dim(H)/2}$ since H contains pairs of vectors $\{p, q\}$ such that $[p,q] = a, [p,t] = 0$ for t not proportional to q and $[q,t] = 0$ for t not proportional to p. The vectors still in H^c will all commute with each other and form an abelian subalgebra of dimension $n - dim(H) - 1$. So $\mathfrak{g} = H_{\dim(H)/2} \oplus ab_{n - \dim(H) - 1}$. ☐

A digression on Lie algebras formed from algebraic groups:

An *algebraic group* over a field $\mathbb{F}$ is a collection of polynomials $\{p_\alpha | \alpha \in J\}$ for some indexing set J on the space of n by n matrices $Mat_n(F)$ such that for any commutative associative unital algebra A over F, the set $G(A) \equiv \{g \in Mat_n(A) | g$ invertible and $p_\alpha(g) = 0$ for all $\alpha \in I\}$ is a group with respect to matrix multiplication.

Examples:

a) If we set $\{p_\alpha\} = \phi$ then the corresponding algebraic group is called the general linear algebraic group and is denoted GL_n. $GL_n(A) = \{$ invertible matrices $a \in Mat_n(A)\}$.

b) Since the determinant of a matrix is a polynomial in the entries of the matrix, we can set $\{p_\alpha\} = \{det - 1\}$. The corresponding algebraic group is known as the special algebraic group, denoted SL_n. $SL_n(A) = \{a \in Mat_n(A) | det\ a = 1\}$.

c) Let $B \in Mat_n(\mathbb{F})$. Define $O_{n,B}(A) \equiv \{a \in GL_n(A) | a^T B a = B\}$.

Exercise 2.4: Prove that $O_{n,B}(A)$ is a subgroup of $GL_n(A)$.

Proof. $O_{n,B} = \{a \in GL_n(A) | a^T B a = B\}$.

Let $a, b \in O_{n,B}$. $a^T B a = B, b^T B b = B \rightarrow (ab)^T B (ab) = b^T a^T B a b = b^T B b = B \rightarrow ab \in O_{n,B}$. $a^T B a = B \rightarrow (a^{-1})^T B a^{-1} = B \rightarrow a^{-1} \in O_{n,B}$. ☐

If B is non-degenerate and symmetric (respectively, skew-symmetric) then $O_{n,B}(A)$ is called the orthogonal (respectively, symplectic) algebraic group.

Let $D = \mathbb{F}[\epsilon]/(\epsilon^2) = \{a + b\epsilon | a, b \in \mathbb{F}, \epsilon^2 = 0\}$. D is called the algebra of dual numbers. Multiplication in D is carried out as follows:

$$(a + b\epsilon)(c + d\epsilon) = ac + (bc + ad)\epsilon$$

The *Lie algebra of an algebraic group* G, denoted $\mathrm{Lie}\, G$, is a subalgebra $\{X \in \mathfrak{gl}_n(\mathbb{F}) | I + \epsilon X \in G(D)\}$.

Theorem 2.5: Lie G is a subalgebra of the Lie algebra $\mathfrak{gl}_n(\mathbb{F})$.

Proof. $X \in \text{Lie}\, G \iff p_\alpha(I+\epsilon X)=0$ for all α and $I+\epsilon X$ is invertible in $Mat_n(D)$.
$(I+\epsilon X)^{-1} = I - \epsilon X$ since $\epsilon^2 = 0 \rightarrow I+\epsilon X$ invertible. Also, $p_\alpha(I)=0$ since $G(D)$ is a group. We can Taylor expand the constraints for elements of the algebraic group as follows:

$$0 = p_\alpha(I+\epsilon X) = p_\alpha(I) + \sum_{i,j} \frac{\partial p_\alpha}{\partial X_{ij}}(I)\epsilon X_{ij}(\epsilon^2=0)$$

Hence $X \in \text{Lie}\, G$ if and only if

$$\sum_{i,j} \frac{\partial p_\alpha}{\partial X_{ij}}(I) X_{ij} = 0, \alpha \in J.$$

Thus Lie G is a subspace of $Mat_n(\mathbb{F})$. If $X, Y \in \text{Lie}\, G$, consider two elements in $G(D)$: $1+\epsilon X \in G(\mathbb{F}[\epsilon]/(\epsilon^2))$ and $1+\epsilon' Y \in G(\mathbb{F}[\epsilon']/(\epsilon'^2))$. Both of these are in $G(\mathbb{F}[\epsilon,\epsilon']/(\epsilon^2,\epsilon'^2))$ and we can consider an equation in this group:

$$\begin{aligned}(1+\epsilon X)(1+\epsilon' Y)(1+\epsilon X)^{-1}(1+\epsilon' Y)^{-1} &= \quad (1+\epsilon X)(1+\epsilon' Y)(1-\epsilon X)(1-\epsilon Y) \\ &= \quad 1+\epsilon\epsilon'(XY-YX) \in G(\mathbb{F}[\epsilon\epsilon']/(\epsilon\epsilon')^2)\end{aligned}$$

which is isomorphic again to the algebra of dual numbers. So Lie G is a subalgebra and we are finished. □

Examples:

a) If $G = GL_n(\mathbb{F})$, then Lie G $= \mathfrak{gl}_n(\mathbb{F})$.
b) If $G = SL_n(\mathbb{F})$, then Lie G $= \{X | det(I+\epsilon X)=1\}$. By writing out the determinant we can see that:

$$\begin{vmatrix} 1+\epsilon x_{11} & \epsilon x_{12} & \cdots & \epsilon x_{1n} \\ \epsilon x_{21} & 1+\epsilon x_{12} & \cdots & \epsilon x_{2n} \\ \vdots & & \ddots & \\ \epsilon x_{n1} & \epsilon x_{n2} & \cdots & 1+\epsilon x_{nn} \end{vmatrix} = 0$$

$= 1+\epsilon(x_{11}+x_{22}+\cdots+x_{nn}) + O(\epsilon^2) = 1+\epsilon tr X \rightarrow tr X = 0.$
So, Lie $SL_n(\mathbb{F}) = \mathfrak{sl}_n(\mathbb{F})$.

Exercise 2.6: Lie $O_{n,B} = \mathfrak{o}_{n,B}(\mathbb{F})$.

Proof. Let $G = O_{n,B}$. $X \in \text{Lie}\, G \rightarrow (1+\epsilon X)^T B(1+\epsilon X) = B \rightarrow B+\epsilon(X^T B + BX) = B \rightarrow X^T B + BX = 0$. □

Next we prepare to prove Engel's theorem by establishing the following definition and lemma:

Definition: An operator $A \in End\ V$ is *nilpotent* if $A^N = 0$ for some positive integer N.

Lemma: If A is nilpotent (on V), then ad A is nilpotent on gl_V.

Proof. Note ad $A = L_A - R_A$, where L_A and R_A are the operations of left and right multiplication by A (in the space of endomorphisms of V) respectively. Since A is nilpotent, L_A and R_A raised to some power N are each zero for large enough N. Moreover, L_A and R_B commute since matrix composition in the space of endomorphisms of V is associative. $(\text{ad}\, A)^{2N} = \sum_{i=0}^{2N} \binom{2N}{i} L_A^{2N-i} R_A^i$, and for all i, $2N-i \geq N$ or $i \geq N \rightarrow L_A^{2N-i} = 0$ or $R_A^i = 0 \rightarrow (\text{ad}\, A)^{2N} = 0$. □