18.745: Lecture 21

Professor: Victor Kač
Scribe: Ruochuan Liu, Zhongtao Wu

Let $(V,\triangle)$ be a root system. Define a reflection $r_\alpha \in$End(V) for every $\alpha \in \triangle$ by:

$$r_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha$$

r_α has properties:

(i) r_α fixes pointwise the hyperplane $\tau_\alpha = \{u \in V | (u,\alpha) = 0\}$.

(ii) $r_\alpha = -\alpha$.

(iii) $r_\alpha^2 = 1$ and $\gamma_\alpha \in O(V, (\cdot,\cdot))$, i.e r_α is invertible and $(\gamma_\alpha(v), \gamma_\alpha(v)) = (v,v)$.

(iv) $r_\alpha(\triangle) = \triangle$.

(i), (ii) and (iii) are obvious. (iv) follows from the string property.

Exercise 21.1 (optional). Show that property (iv) along with $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$ implies the string property.

Proof. Denote $<\beta,\alpha>:= \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. Then $<\beta,\alpha>\in \mathbb{Z}$ for all $\alpha,\beta \in \Delta$, and $r_\alpha(\beta) = \beta - <\beta,\alpha>\alpha$.

Step I: Let α, β be nonproportional roots. If $(\alpha,\beta) > 0$, then $\alpha - \beta$ is a root. If $(\alpha,\beta) < 0$, then $\alpha + \beta$ is a root.

Proof: If $(\alpha,\beta) > 0$, then both $<\alpha,\beta>$ and $<\beta,\alpha>$ are positive. By Cauchy's inequality,

$$<\alpha,\beta><\beta,\alpha>= \frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} < 4,$$

where the inequality is strict since α and β are nonproportional roots. Hence, at least one of $<\alpha,\beta>$, $<\beta,\alpha>$ equals 1. If $<\alpha,\beta>= 1$, then $r_\beta(\alpha) = \alpha - \beta \in \Delta$; similarly, if $<\beta,\alpha>= 1$, then $\beta - \alpha \in \Delta$, hence $r_{\beta-\alpha}(\beta - \alpha) = \alpha - \beta \in \Delta$.

The case $(\alpha,\beta) < 0$ is similar.

Step II: The α-string through β is unbroken, i.e., if $p, q \in \mathbb{Z}_+$ are the largest integers for which $\beta - p\alpha, \beta + q\alpha \in \Delta$, then $\beta + i\alpha \in \Delta$, $\forall -p \le i \le q$.

Proof: If not, we can find $-p \le i < j \le q$ such that $\beta + i\alpha \in \Delta$, $\beta + (i+1)\alpha \notin \Delta$, $\beta + (j-1)\alpha \notin \Delta$, $\beta + j\alpha \in \Delta$. But then the claim in Step I implies both $(\beta + i\alpha, \alpha) \ge 0$, $(\beta + j\alpha, \alpha) \le 0$. Hence, $(\alpha,\alpha) \le 0$. Contradiction!

Step III: p, q as in Step II, then $p - q =<\beta,\alpha>$.

Proof: Since r_α just adds or subtracts a multiple of α to any root, the string is invariant under r_α. In particular, $r_\alpha(\beta + q\alpha) = \beta - p\alpha$. The left hand side is also equal to $\beta - <\beta,\alpha>\alpha - q\alpha$. Hence, $p - q =<\beta,\alpha>$.

Thus, we proved the string property for Δ. $\square$

Definition 1. The *Weyl group* of a root system is a subgroup W of $O(V, (\cdot, \cdot))$ generated by all reflections γ_α for $\alpha \in \triangle$. This is a finite group since it permutes elements of a finite set which spans V.

Example 2. $\triangle_{A_r} = \{\varepsilon_i - \varepsilon_j | 1 \leq i, j \leq r+1, i \neq j\}$. Let $\alpha = \varepsilon_i - \varepsilon_j$, then

$$r_\alpha(\varepsilon_k) = \left\{ \begin{array}{ccc} \varepsilon_k & : & k \neq i, j \\ \varepsilon_j & : & k = i \\ \varepsilon_i & : & k = j \end{array} \right\}$$

Hence $r_\alpha = (ij)$, i.e., it just permutes ε_i and ε_j. Thus the Weyl group of A_r is S_{r+1}.

Exercise 21.2. Compute the Weyl group of the root systems B, C and D.

Proof.

(a) Root system $\Delta_{B_r} = \{\pm\epsilon_i \pm \epsilon_j \ (i \neq j),\ \pm\epsilon_i\}$.

- When $\alpha = \epsilon_i - \epsilon_j$, $r_\alpha(\epsilon_k) = \begin{cases} \epsilon_k & \text{if } k \neq i, j; \\ \epsilon_j & \text{if } k = i; \\ \epsilon_i & \text{if } k = j. \end{cases}$
- When $\alpha = \epsilon_i + \epsilon_j$, $r_\alpha(\epsilon_k) = \begin{cases} \epsilon_k & \text{if } k \neq i, j; \\ -\epsilon_j & \text{if } k = i; \\ -\epsilon_i & \text{if } k = j. \end{cases}$
- When $\alpha = \epsilon_i$, $r_\alpha(\epsilon_k) = \begin{cases} \epsilon_k & \text{if } k \neq i; \\ -\epsilon_i & \text{if } k = i. \end{cases}$

Hence, the Weyl group is generated by all permutations of the set $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_r\}$ and the operations of sign changes. In terms of group, it is isomorphic to $S_r \times \mathbb{Z}_2^r$.

(b) Root system $\Delta_{C_r} = \{\pm\epsilon_i \pm \epsilon_j \ (i \neq j),\ \pm 2\epsilon_i\}$.

We can check that $r_\alpha(\epsilon_k)$ has exactly the same form as the previous case, hence its Weyl group is also $S_r \ltimes \mathbb{Z}_2^r$.

(c) Root system $\Delta_{D_r} = \{\pm\epsilon_i \pm \epsilon_j,\ i \neq j\}$.

In this case, the Weyl group is generated by the first two types of reflections in (a). Also, notice that

$$r_{\epsilon_i - \epsilon_j} \circ r_{\epsilon_i + \epsilon_j}(\epsilon_k) = \begin{cases} \epsilon_k & \text{if } k \neq i, j; \\ -\epsilon_i & \text{if } k = i; \\ -\epsilon_j & \text{if } k = j. \end{cases}$$

This is the same as changing signs for a pair (ϵ_i, ϵ_j). Hence, each element in the Weyl group is a permutation of set $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_r\}$, composed with an even number of sign changes. In terms of group, it is isomorphic to $S_r \ltimes \mathbb{Z}_2^{r-1}$.

$\square$

Exercise 21.3. If $A \in O(V, (\cdot, \cdot))$, then $A r_\alpha A^{-1} = r_{A(\alpha)}$.

Proof. Let $AT_\alpha := \{A(u) | (\alpha, u) = 0\}$. Since $A \in O(V, (\cdot,\cdot))$, we have $(A(u), A(\alpha)) = (u, \alpha) = 0$ for all $A(u) \in AT_\alpha$. Hence $r_{A(\alpha)}(AT_\alpha) = AT_\alpha$.

Also, $Ar_\alpha A^{-1}(AT_\alpha) = Ar_\alpha(T_\alpha) = AT_\alpha$. So both reflections $Ar_\alpha A^{-1}$ and $r_{A(\alpha)}$ fix the hyperplane AT_α, hence $Ar_\alpha A^{-1} = r_{A(\alpha)}$.

□

Recall that, given a choice of $f \in V^*$ such that $f(\alpha) \neq 0$ for all $\alpha \in \triangle$, we get a subset $\triangle_+ = \{\alpha \in \triangle | f(\alpha) > 0\}$ and $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ of simple roots such that any $\alpha \in \triangle_+$ is of the form $\alpha = \Sigma_i k_i \alpha_i$, $k_i \in \mathbb{Z}_{\geq 0}$. The reflections $s_i = r_{\alpha_i}$ are called *simple reflections*. We use the notation $\text{height}(\alpha) := \sum_i k_i$.

Theorem 3.

(a) $\triangle_+ \backslash \{\alpha_i\}$ *is* s_i*-invariant.*

(b) If $\alpha \in \triangle_+ \backslash \Pi$, *there is a* s_i *such that height*$(s_i(\alpha))$ <*height*(α).

(c) If $\alpha \in \triangle_+ \backslash \Pi$, *we can choose a sequence of simple reflections* $s_{i_1}, \ldots, s_{i_k}$ *such that* $s_{i_1} \ldots s_{i_k}(\alpha) \in \Pi$ *and* $s_{i_j} \ldots s_{i_k}(\alpha) \in \triangle_+$ *for each* $1 \leq j \leq k$.

(d) W *is generated by simple reflections.*

Proof.

(a) Applying simple reflection s_i changes sign of at most one coefficient k_i of $\alpha \in \triangle_+$. If k_i changes to negative, then $s_i(\alpha)$ wouldn't be a root, hence $s_i(\alpha) \in \triangle_+$.

(b) If height$(s_i(\alpha))$ doesn't decrease for all s_i, then from $s_i(\alpha) = \alpha - \frac{2(\alpha,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$ we get $(\alpha, \alpha_i) \leq 0$ for all i. Hence $(\alpha, \alpha) = \Sigma_i k_i(\alpha, \alpha_i) \leq 0$, a contradiction!

(c) follows from (b) and (a).

(d) Denote W' the subgroup of W generated by simple reflections. By (c), for any $\alpha \in \triangle_+$, there exists $w \in W'$ such that $w(\alpha) = \alpha_i \in \Pi$ for some i. By Ex21.3, $wr_\alpha w^{-1} = r_{w(\alpha)}$, hence $r_\alpha = w^{-1} s_i w \in W'$.

□

Consider $V \backslash \bigcup_{\alpha \in \triangle_+} T_\alpha = \coprod_j C_j$, where C_j are connected components of this set. C_j's are called open chambers, $\overline{C_j}$'s are called (closed) chambers. Also, define the *fundamental chamber*: $\overline{C} = \{v \in V | (\alpha_i, v) \geq 0$ $i = 1, \ldots, r\}$.

Exercise 21.4. Show that $T_\alpha \cap C = \varnothing$, where $C = \{v \in V | (\alpha_i, v) > 0, i = 1, \ldots, r\}$ is the open fundamental chamber. Hence the fundamental chamber is a chamber.

Proof. Suppose $v \in T_\alpha \cap C$, then $(v, \alpha) = 0$ and $(v, \alpha_i) > 0$ for $i = 1, \ldots, r$. But $\alpha = \sum_{i=1}^r k_i \alpha_i$, where $k_i \in \mathbb{Z}_+$. Hence, $k_i = 0$ for all i, and consequently $v = 0$.

So we proved $T_\alpha \cap C = \varnothing$. Since C is connected, $C \subset C_j$ for some j. On the other hand, $(\alpha_i, v) \neq 0$, $\forall i, v \in C_j$ by definition. And since the inner product is a continuous function of v, we conclude that $(\alpha_i, v) > 0$ $\forall v \in C_j$. Hence $C_j \subset C$.

Thus, $\bar{C} = \bar{C_j}$, i.e, the fundamental chamber is a chamber.

□

Theorem 4.

(a) W *permutes all chambers transitively, i.e for any chamber* $\overline{C_1}$ *and* $\overline{C_2}$, *there exists* $w \in W$ *such that* $w\overline{C_1} = \overline{C_2}$.

(b) Let $\triangle_+$ *and* $\triangle'_+$ *be subsets of positive roots of* $\triangle$ *defined by* f *and* f' *respectively . Then there exists* $w \in W$ *such that* $w(\triangle_+) = \triangle'_+$. *In particular, the Cartan matrix is independent of the choice of* f.

Proof.

(a) Choose $P_i \in C_i$ $(i = 1, 2)$ such that the interval $[P_1, P_2]$ doesn't intersect any of $\tau_\alpha \cap \tau_\beta$, where α, $\beta \in \triangle_+$ and $\alpha \neq \beta$. Now move along the interval $[P_1, P_2]$ until we hit a hyperplane τ_α. Apply reflection r_α to $\overline{C_1}$. Keep moving and applying reflections until we reach $\overline{C_2}$.

(b) $\triangle_+$ and $\triangle_+^{'}$ define fundamental chamber $\overline{C}$ and $\overline{C'}$ respectively. By (a), there exists $w \in W$ such that $w(\overline{C}) = \overline{C'}$. Hence $w(\triangle_+) = w(\triangle_+^{'})$, since $\overline{C} = \{v \in V | (\alpha_i, v) \geq 0 \ i = 1, \ldots, r\}$.

□

Definition 5. Fix $\Pi \subset \triangle_+ \subset \triangle$, then we have simple reflections $s_1, \ldots, s_r \in W$. Given $w \in W$, an expression $w = s_{i_1} \ldots s_{i_l}$ is called *reduced* if l is minimal possible. We let $l = l(w)$ called the length of w. Note that $\det w = (-1)^{l(w)}$ since $\det s_i = -1$.

Example 6. $l(s_i) = 1$, $l(s_i s_j) = 2$ if $i \neq j$, but $l(s_i^2) = 0$.

Lemma 7 (Exchange Lemma). *Suppose that $s_{i_1} \ldots s_{i_{t-1}}(\alpha_{i_t}) \in \triangle_-$, then the expression $w = s_{i_1} \ldots s_{i_t}$ is not reduced. Namely, there exists $1 \leq r < t$ such that $w = s_{i_1} \ldots s_{i_{r-1}} s_{i_{r+1}} \ldots s_{i_{t-1}}$.*

Proof. Consider the roots $\beta_k = s_{i_{k+1}} \ldots s_{i_{t-1}}(\alpha_{i_t})$ for $0 \leq k \leq t-1$. Then $\beta_0 \in \triangle_-$ by assumption and $\beta_{t-1} = \alpha_{i_t} \in \triangle_+$. Hence there exists $0 \leq r \leq t-1$ such that $\beta_{r-1} \in \triangle_-$ and $\beta_r \in \triangle_+$. But by definition $\beta_r = s_{i_r}(\beta_r)$, hence $\beta_r = \alpha_{i_r}$. Recall that, by definition, $\beta_r = s_{i_{r+1}} \ldots s_{i_{t-1}}(\alpha_{i_t}) = \alpha_{i_r}$. Thus if we denote $\overline{w} = s_{i_{r+1}} \ldots s_{i_{t-1}}$, using Ex21.3 we see that $\overline{w} s_{i_t} \overline{w}^{-1} = s_{i_r}$, thus $\overline{w} s_{i_t} = s_{i_r} \overline{w}$. Now multiplying both sides by $s_{i_1} \ldots s_{i_r}$, we get the result. □

Corollary 8. *W acts simply transitive on chambers, i.e if $w(\overline{C}) = \overline{C}$, then $w = 1$. In particular if $\lambda \in C$(open chamber), $w(\lambda) = \lambda$, then $w = 1$.*

Proof. If $w \neq 1$, take its reduced expression: $w = s_{i_1} \ldots s_{i_l}$. If $w(\overline{C}) = \overline{C}$, then $w(\Pi) = \Pi$, in particular $w(\alpha_{i_l}) \in \triangle_+$. On the other hand, $w(\alpha_{i_l}) = s_{i_1} \ldots s_{i_{l-1}}(-\alpha_{i_l}) \in \triangle_+$ means $s_{i_1} \ldots s_{i_{l-1}}(\alpha_{i_l}) \in \triangle_-$, hence by exchange lemma $s_{i_1} \ldots s_{i_l}$ is not a reduced expression. That's a contradiction! □