Lecture 22 – November 30, 2004

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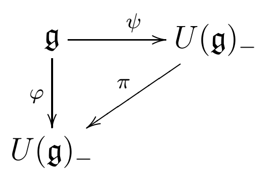
Definition. An enveloping algebra of a Lie algebra $\mathfrak{g}$ is a pair (U, φ) where U is a unital associative algebra and φ is a Lie algebra homomorphism $\mathfrak{g} \to U_-$ (where U_- denotes the Lie algebra structure on U given by $[a, b] = ab - ba$).

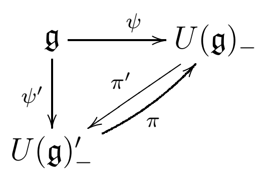
Example. Given a representation $\varphi : \mathfrak{g} \to \text{End } V = U$, we have an enveloping algebra (U, φ) of $\mathfrak{g}$.

Definition. The universal enveloping algebra of $\mathfrak{g}$ is an enveloping algebra $(U(\mathfrak{g}), \varphi)$ which is universal in the sense that for any other enveloping algebra (U, φ), there is a unique associative algebra homomorphism $\pi : U(\mathfrak{g}) \to U$, which makes the following diagram commute: (with respect to Lie algebra homomorphisms)



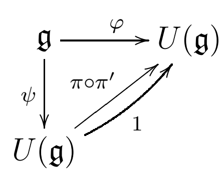
Theorem. For any Lie algebra $\mathfrak{g}$, a universal enveloping algebra exists and is unique.

Proof. 1. Uniqueness: Suppose there are two universal enveloping algebras, $U(\mathfrak{g})_-$ and $U'(\mathfrak{g})_-$.



But then $\pi \circ \pi' : U(\mathfrak{g}) \to U(\mathfrak{g})$, and also $\pi' \circ \pi : U'(\mathfrak{g}) \to U'(\mathfrak{g})$.

But



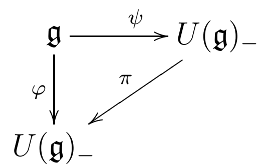
is a commuting diagram.

By uniqueness, we see that $\pi \circ \pi' = 1$, and by symmetry, $\pi' \circ \pi = 1$.

Existence: Let a_i be a basis for $\mathfrak{g}$. Let $U(\mathfrak{g})$ be a unital associative algebra generated by a_i, with relations $a_i a_j - a_j a_i = [a_i, a_j]$. Let $\varphi(a_i) = a_i$.

Since we divided by the above relations, φ is a Lie algebra homomorphism. So, this is an enveloping algebra.

It is universal because:



This diagram commutes if we let $\pi(a_{i_1}...a_{i_s}) = \varphi(a_{i_1})...\varphi(a_{i_s})$.

□

Poincare-Birkhoff-Witt Theorem. The monomials $a_{i_1}...a_{i_s}$ with $i_1 \leq i_2 \leq ...i_s$ form a basis of $U(\mathfrak{g})$. In particular, $\psi : \mathfrak{g} \to U(\mathfrak{g})_-$ is an embedding.

Proof. I) These monomials span $U(\mathfrak{g})$. Of course, the unordered monomials $a_{j_1}..a_{j_s}$ span $U(\mathfrak{g})$. We prove by induction on (s, number of inversions) that using relations $[a_i, a_j] = a_i a_j - a_j a_i$, we can bring this monomial to a linear combination of ordered monomials. If $...a_{j_t} a_{j_t+1}...$ with $j_t > j_t + 1$, we replace $a_{j_t} a_{j_t+1}$ with $[a_{j_t}, a_{j_t+1}] + a_{j_t+1} a_{j_t}$, (where $[a_{j_t}, a_{j_t+1}]$ is a linear combination of the generators a_i). We get a sum of monomials, where the number of factors is less than s in all but the last one, and in that one, the number of inversions drops by 1. Thus we can apply the inductive assumption.

II) Let $\mathfrak{B}_n$ be the free vector space on generators $u_{i_1}...u_{i_n}$, $i_1 \leq i_2... \leq i_n$. Let $\mathfrak{B} = 1 \oplus \mathfrak{B}_1 \oplus \mathfrak{B}_2 \cdots$. We will show that there is a linear map $\sigma : U(\mathfrak{g}) \to \mathfrak{B}$ such that the image of the ordered monomials under σ is linearly independent, which will complete the theorem.

Let us define σ by

$$\sigma(1) = 1, \sigma(a_{i_1} \cdots a_{i_n}) = u_{i_1} \cdots u_{i_n} \text{ if } i_1 \leq \cdots \leq i_n. \tag{1}$$

Finally, we want to have

$$\sigma(a_{j_1} \cdots a_{j_n} - a_{j_1} \cdots a_{j_{k+1}} a_{j_k} \cdots a_{j_n}) = \sigma(a_{j_1} \cdots [a_{j_k}, a_{j_{k+1}}] \cdots a_{j_n}). \tag{2}$$

To show that such a linear map exists, we assume it has been defined for all monomials of degree less than or equal to $n-1$ in $U(\mathfrak{g})$, and show that it can be extended to a map on all monomials of degree less than or equal to n in $U(\mathfrak{g})$. So, assuming σ has been defined for all monomials of degree less than or equal to $n-1$ in $U(\mathfrak{g})$, if $a_{i_1} \cdots a_{i_n}$ is an ordered monomial, let $\sigma(a_{i_1} \cdots a_{i_n}) = u_{i_1} \cdots u_{i_n}$.

If $a_{i_1} \cdots a_{i_n}$ is not ordered, suppose $j_k > j_{k+1}$. Then set

$$\sigma(a_{j_1} \cdots a_{j_n}) = \sigma(a_{j_1} \cdots a_{j_{k+1}} a_{j_k} \cdots a_{j_n}) + \sigma(a_{j_1} \cdots [a_{j_k}, a_{j_{k+1}}] \cdots a_{j_n}). \tag{3}$$

We must check that this map is well defined, in that its independent of the choice of the pair (j_k, j_{k+1}). Suppose (j_l, j_{l+1}) is another pair with $j_l > j_{l+1}$. There are two cases: 1) $l > k+1$ and 2) $l = k+1$.

1) Set $a_{j_k} = u, a_{j_{k+1}} = v, a_{j_l} = w, a_{j_{l+1}} = x$. Then the induction hypothesis permits us to write for the right hand side of (3)

$$\sigma(\cdots vu \cdots xw \cdots)+\sigma(\cdots vu \ldots [wx] \cdots)+\sigma(\cdots [uv] \ldots xw \cdots)+\sigma(\cdots [uv] \ldots [wx] \cdots) \quad (4)$$

If we start with (j_l, j_{l+1}), we obtain

$$\begin{aligned} \sigma(\cdots uv \cdots xw \cdots) + \sigma(\cdots uv \ldots [wx] \cdots) = \sigma(\cdots vu \ldots xw \cdots)+ \\ + \sigma(\cdots [uv] \ldots xw \cdots) + \sigma(\cdots vu \cdots [wx] \cdots) + \sigma(\cdots [uv] \ldots [wx] \cdots) \end{aligned} \quad (5)$$

This is the same as the value obtained before.

2) Set $a_{j_k} = u, a_{j_{k+1}} = v, a_{j_{l+1}} = w$. Using the induction hypothesis, the right hand side of (3) becomes

$$\sigma(\cdots wvu \cdots) + \sigma(\cdots [vw]u \cdots) + \sigma(\cdots v[uw] \cdots) + \sigma(\cdots [uv]w \cdots) \quad (6)$$

Similarly, if we start with $\sigma(\cdots wvu \cdots) + \sigma(\cdots u[vw])$, we can wind up with

$$\sigma(\cdots wvu \cdots) + \sigma(\cdots w[uv] \cdots) + \sigma(\cdots [uw]v \cdots) + \sigma(\cdots u[vw] \cdots) \quad (7)$$

So we must show that σ applied to

$$\begin{aligned} (\cdots [vw]u \cdots) - (\cdots u[vw] \cdots) + (\cdots v[uw] \cdots)- \\ - (\cdots [uw]v \cdots) + (\cdots [uv]w \cdots) - (\cdots w[uv] \cdots), \end{aligned} \quad (8)$$

a monomial of degree less than or equal to $n-1$, gives 0.

But from the properties of σ on all monomials of degree less than or equal to $n-1$ in $U(\mathfrak{g})$, if $(\cdots xy \cdots)$ is a monomial of degree less than or equal to $n-1$,

$$\sigma(\cdots xy \cdots) - \sigma(\cdots yx \cdots) - \sigma(\cdots [xy] \cdots) = 0. \quad (9)$$

Hence σ applied to (8) gives

$$(\cdots [[vw]u] \cdots) + (\cdots [v[uw]] \cdots) + (\cdots [[uv]w] \cdots) \quad (10)$$

which is zero by the Jacobi identity and linearity of σ. Thus σ is well defined on monomials of degree less than or equal to n, as well, and we extend σ linearly to the space spanned by all monomials of degree less than or equal to n in $U(\mathfrak{g})$. In this way, σ is defined on all of $U(\mathfrak{g})$, and clearly, the image of the ordered monomials in $U(\mathfrak{g})$ is linearly independent in $\mathfrak{B}$, as they are in bijection with the generators of $\mathfrak{B}$. Thus the proof is completed.

□

Remark. Any representation π of a Lie algebra $\mathfrak{g}$ in a vector space V extends to a represenation of $U(\mathfrak{g}) \to \text{End } V$ (as associative algebras).

Definition. Given a representation π of $\mathfrak{g}$ over V, V can be considered a $\mathfrak{g}$-module by defining a binary product $\mathfrak{g} \times V$ into V mapping $g \cdot v$ to $\pi(\mathfrak{g})v$. Thus the defining property of a $\mathfrak{g}$-module is: $[a, b]v = abv - bav$. By a homomorphism of $\mathfrak{g}$-modules we mean a linear map $\varphi : V_1 \to V_2$ such that $\varphi(gv) = g\varphi(v)$. An isomorphism is a homomorphism φ which is bijective.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a fixed non-degenerate invariant symmetric bilinear form $(.,.)$. Chose a basis u_i of $\mathfrak{g}$ and let v_i be the dual basis, which means $(u_i, v_j) = \delta_{ij}$.

Definition. The Casimir operator $\Omega = \sum_i u_i v_i \in U(\mathfrak{g})$.

Exercise 22.1. *Ω is independent of the choice of the basis u_i.*

Solution. Take another basis $u_i{}'$ with dual basis $v_i{}'$. Since u_i was a basis, we can write $u_i{}' = \sum_j a_{ij} u_j$. In this basis, $\Omega' = \sum_i u_i{}' v_i{}'$.

Note that by definition, $(u_i{}', v_k{}') = \delta_{ij} = (\sum_j a_{ij} u_j, v_k') = \sum_j (a_{ij})(u_j, v_k')$. Let the matrix $A = \langle a_{ij} \rangle$, and $B = \langle (u_j, v_k') \rangle$. Clearly, $AB = I$. Now consider $I = BA = \sum_k (u_j, v_k') a_{ki}$. This implies that $\sum_k (u_j v_k') a_{ki} = \delta_{ji} = (u_j, \sum_k a_{ki} v_k')$. But since v_i is the unique vector such that $(u_j, v_i) = \delta_{ji}$, it follows that $v_i = \sum_k a_{ki} v_k'$.

Finally, $\Omega' = \sum_i u_i{}' v_i{}' = \sum_i (\sum_j a_{ij} u_j) v_i{}' = \sum_j \sum_i a_{ij} v_i{}' u_j$. By the result of the previous paragraph, this equals $\sum_j v_j u_j$, which gives the desired result, $\Omega' = \sum_i u_i v_i = \Omega$.

□

Lemma 1. (on dual bases) For $a \in \mathfrak{g}$ write $[a, u_i] = \sum_j a_{ij} u_j$ and $[b, u_i] = \sum_j b_{ij} u_j$. Then $a_{ij} = -b_{ji}$ (under the above assumptions).

Proof. We have: $([a, u_i], v_k) = \sum_j a_{ij}(u_j, v_k) = a_{ik}$ and similarly $([a, v_i], u_k) = b_{ik}$. Hence $a_{ik} = (a, [u_i, v_k])$ and $b_{ik} = (a, [v_i, u_k])$. Therefore $a_{ik} = -b_{ki}$. □

Definition. Let V be a $\mathfrak{g}$-module, where $\mathfrak{g}$ is a Lie algebra. A 1-cocycle is a linear map $\varphi : \mathfrak{g} \mapsto V$ such that $\varphi([a, b]) = a\varphi(b) - b\varphi(a)$. The space of 1-cocycles is denoted $Z(\mathfrak{g}, V)$.

Example of a 1-cocycle. The trivial 1-cocycle associated to $v \in V$ is $\varphi_v(a) = a \cdot v$.

Exercise 22.2. *Check that φ_v is a 1-cocycle.*

Solution. $\varphi_v([a, b]) = [a, b]v = a(bv) - b(av) = a\varphi_v(b) - b\varphi_v(a)$ □

The trivial 1-cocycles form a subspace $B(\mathfrak{g}, V)$ of $Z(\mathfrak{g}, V)$. Let $H^1(\mathfrak{g}, V) = Z(\mathfrak{g}, V)/B(\mathfrak{g}, V)$.

Main Theorem on Cohomology. If $\mathfrak{g}$ is a semi-simple Lie algebra over an algebraicly closed field of characteristic 0, and V is a finite dimensional $\mathfrak{g}$-module, then $H^1(\mathfrak{g}, V) = 0$, i.e., every 1-cocycle is trivial.

Exercise 22.3. *$H^1(\mathfrak{g}, V_1 \oplus V_2) = H^1(\mathfrak{g}, V_1) \oplus H^1(\mathfrak{g}, V_2)$ where $V_1 \oplus V_2$ denotes the direct sum of $\mathfrak{g}$-modules V_1 and V_2.*

Solution. First we show that $Z(\mathfrak{g}, V_1 \oplus V_2) = Z(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)$. It is clear that $Z(\mathfrak{g}, V_1 \oplus V_2) \supset Z(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)$. Furthermore, every 1-cocycle $\varphi \in Z(\mathfrak{g}, V_1 \oplus V_2)$ can be decomposed as $\pi_1 \circ \varphi \oplus \pi_2 \circ \varphi \in Z(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)$.

It is also clear that $B(\mathfrak{g}, V_1 \oplus V_2) = B(\mathfrak{g}, V_1) \oplus B(\mathfrak{g}, V_2)$ since $\varphi_{v_1 \oplus v_2} = \varphi_{v_1} \oplus \varphi_{v_2}$.

Therefore $H^1(\mathfrak{g}, V_1 \oplus V_2) = Z(\mathfrak{g}, V_1 \oplus V_2)/B(\mathfrak{g}, V_1 \oplus V_2) = Z(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)/B(\mathfrak{g}, V_1) \oplus B(\mathfrak{g}, V_2) = Z(\mathfrak{g}, V_1)/B(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)/B(\mathfrak{g}, V_2) = H^1(\mathfrak{g}, V_1) \oplus H^1(\mathfrak{g}, V_2)$. □

Lemma 2. In the situation of lemma 1. Let V be a $\mathfrak{g}$-module and $\varphi : \mathfrak{g} \mapsto V$ a 1-cocycle. Then for any $a \in \mathfrak{g}$ we have: $a \sum_i u_i \varphi(v_i) = \Omega\varphi(a)$.

Proof. $a \sum_i u_i \varphi(v_i) = \sum_i [a, u_i]\varphi(v_i) + \sum_i u_i a\varphi(v_i) = \sum_{i,j} a_{ij} u_j \varphi(v_i) + \sum_i u_i a\varphi(v_i) = \sum_j u_j \varphi(\sum_i a_{ij} v_i) + \sum_i u_i a\varphi(v_i)$. Now $\sum_i a_{ij} v_i = -\sum_i b_{ji} v_i = -[a, v_i]$, so $a \sum_i u_i \varphi(v_i) = -\sum_j u_j \varphi([a, v_i]) + \sum_j u_j a\varphi(v_j) = \sum_j u_j (a\varphi(v_j) - \varphi([a, v_i])) = \sum_j u_j v_j \varphi(a) = \Omega\varphi(a)$. □

Corollary. $\mathfrak{g}$ commutes with Ω, i.e., in any $\mathfrak{g}$-module $a(\Omega v) = \Omega(av)$ for all $a \in \mathfrak{g}$, $v \in V$.

Proof. Apply lemma 2 to the trivial cocycle $\varphi_v(a) = a \cdot v$: $a(\Omega v) = a \sum_i u_i v_i(v) = \Omega(av)$. □

Proof of the Main Theorem on Cohomology. The proof is by induction on the dimension of the $\mathfrak{g}$-module V.

First note that we may assume that V is faithful, that is, $aV = 0$ implies $a = 0$ for $a \in \mathfrak{g}$. Indeed let $\mathfrak{g}_0 = \{a \in \mathfrak{g} | aV = 0\}$. This is an ideal of $\mathfrak{g}$. Hence $\mathfrak{g}_0$ and $\mathfrak{g}/\mathfrak{g}_0$ are again semi-simple Lie algebras. In particular $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{g}_0$. Let φ be a 1-cocycle of $\mathfrak{g}$ in V, i.e., $\varphi([a, b]) = a\varphi(b) - b\varphi(a)$. If $a, b \in \mathfrak{g}_0$, we get $\varphi([a, b]) = 0$. So $\varphi([\mathfrak{g}_0, \mathfrak{g}_0]) = 0$, therefore $\varphi([a, b]) = 0$. Hence $\varphi : \mathfrak{g}/\mathfrak{g}_0 \mapsto V$, so we may replace $\mathfrak{g}$ by $\mathfrak{g}/\mathfrak{g}_0$.

We want to apply lemma 2.

Take $(a, b) = \mathrm{tr}_V ab$. It is non-degenerate since $\mathfrak{g}$ is semi-simple. Let $\{u_i\}$ be a basis of $\mathfrak{g}$, $\{v_i\}$ the dual basis, and $\Omega = \sum_i u_i v_i$ the Casmir operator. We decompose $V = V_0 \oplus V_1$, where V_0 is the generalized eigenspace of Ω attached to 0 and V_1 is the sum of all the other generalized eigenspaces. By the corollary V_0 and V_1 are $\mathfrak{g}$-invariant. So by Exercise 22.3 $H^1(\mathfrak{g}, V) = H^1(\mathfrak{g}, V_0) \oplus H^1(\mathfrak{g}, V_1)$. If V_0 and V_1 are not both zero, by the induction hypothesis $H^1(\mathfrak{g}, V_0) = 0$ and $H^1(\mathfrak{g}, V_1) = 0$ and so $H^1(\mathfrak{g}, V) = 0$. Hence we may assume $V = V_0$ or V_1.

Case 1: $V = V_1$. So Ω is invertible. Let $v = \sum_i u_i \varphi(v_i)$. Lemma 2 now states that $a(v) = \Omega\varphi(a)$. Hence $\varphi(a) = \Omega^{-1}a(v) = a(\Omega^{-1}v)$. So $\varphi = \varphi_{\Omega^{-1}v}$ is a trivial cocycle.

Case 2: $V = V_0$. So Ω is a nilpotent operator. Hence $\mathrm{tr}_V(\Omega) = 0$, but $\mathrm{tr}_V(\Omega) = \mathrm{tr}_V \sum_i u_i v_i = \sum_i (u_i, v_i) = \dim \mathfrak{g}$. So $\mathfrak{g} = 0$ and $H^1(\mathfrak{g}, V) = 0$. $\square$