18.745: LECTURE 23

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From now on, let $\mathfrak{g}$ be an arbitrary finite-dimensional Lie Algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0.

Theorem. *(Weyl Complete Reducibility) If $\mathfrak{g}$ is semisimple, then any finite-dimensional $\mathfrak{g}$-module V is completely reducible, ie: for any submodule U there exists a complementary submodule U' so that $V = U \oplus U'$.*

Corollary. *Any finite-dimensional $\mathfrak{g}$-module is isomorphic to a direct sum of irreducible $\mathfrak{g}$-modules.*

Theorem. *(Levi) Let $\mathfrak{g}$ be any finite-dimensional Lie Algebra over $\mathbb{F}$, and $R(\mathfrak{g})$ the radical of $\mathfrak{g}$. Then there exists a semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + R(\mathfrak{g})$. (As previously noted, $\mathfrak{s} \cap R(\mathfrak{g}) = \{0\}$).*

Theorem. *(Mal'cev) In Levi's theorem, if $\mathfrak{s}_1$ is another semisimple subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s}_1 + R(\mathfrak{g})$, then there exists an automorphism ϕ of $\mathfrak{g}$ such that $\phi(\mathfrak{s}) = \mathfrak{s}_1$.*

Recall that a "projector" of a vector space V to a subspace $U \subset V$ is a linear operator P on V such that $P(V) = U$ and $P(u) = u$ for all $u \in U$. Note that $V = U \oplus Ker(P)$ for any projector P.

Given a projector P_0 of V to U, any other projector of V to U is of the form $P = P_0 + A$ where $A(V) \subset U$ and $A(U) = 0$.

Proof. (**Weyl's Theorem**) Consider $End\ V$ as a $\mathfrak{g}$-module, with: $a * A = [a, A] = aA - Aa$, for $a \in \mathfrak{g}$, $A \in End\ V$. Notational Remark: since V is a $\mathfrak{g}$-module, we are identifying elements of $\mathfrak{g}$ with elements of $End\ V$. Hence, "aA" and "Aa" represent composition in $End\ V$, while $a * A$ represents the $\mathfrak{g}$-module action defined above.

Let M be the subspace of $End\ V$ consisting of all A such that $A(V) \subset U$ and $A(U) = 0$. This is a submodule of $End\ V$:

$$(a * A)V = (aA - Aa)V \subset a(A(V)) + A(a(V)) \subset a(U) + A(V) \subset U$$

$$(a * A)U = (aA - Aa)U \subset a(A(U)) + A(a(U)) \subset a(0) + A(U) = 0$$

Pick an arbitrary projector $P_0 : V \longrightarrow U$, and define the following 1-cocycle:

$$f : \mathfrak{g} \longrightarrow M; f(a) = a * P_0 = aP_0 - P_0 a$$

It is clear f maps into M since a and P_0 commute on U, and:

$$(aP_0 - P_0 a)V \subset a(P_0(V)) + P_0(a(V)) \subset a(U) + P_0(V) \subset U$$

f is also clearly a 1-cocycle, since:

$$f([a, b]) = [a, b] * P_0 = a * (b * P_0) - b * (a * P_0) = a * f(b) - b * f(a).$$

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(the second equality is just the defining property of a $\mathfrak{g}$-module)

Thus, by the main theorem of Lecture 22, f is trivial: there exists $A \in M$ such that $f(a) = a * A$ for all $a \in \mathfrak{g}$. Equivalently: $a * P_0 = a * A$, or $a * (P_0 - A) = 0$. Let $P = P_0 - A$. Observe that P is just a projector of V to U, and we have $a * P = 0$, ie, $aP = Pa$, for all $a \in \mathfrak{g}$. Hence, $Ker(P)$ is a-invariant for all $a \in \mathfrak{g}$, and thus $Ker(P)$ is a $\mathfrak{g}$-submodule of V. Since $V = U \oplus Ker(P)$, $Ker(P)$ is a $\mathfrak{g}$-submodule of V complementary to U, as desired. □

Proof. (**Levi's Theorem**, using induction on dim $\mathfrak{g}$) If $Rad(\mathfrak{g})$ is not abelian, consider $\bar{\mathfrak{g}} = \mathfrak{g}/[Rad(\mathfrak{g}), Rad(\mathfrak{g})]$. Since $dim(\bar{\mathfrak{g}}) < dim(\mathfrak{g})$, by the inductive assumption there exists semisimple $\bar{\mathfrak{s}} \subset \bar{\mathfrak{g}}$ such that $\bar{\mathfrak{g}} = \bar{\mathfrak{s}} + Rad(\bar{\mathfrak{g}})$. Hence $\mathfrak{g} = \mathfrak{g}_1 + Rad(\mathfrak{g})$, where $\mathfrak{g}_1$ is the preimage of $\bar{\mathfrak{s}}$ in $\mathfrak{g}$, and $dim(\mathfrak{g}_1) < dim(\mathfrak{g})$. Now apply the inductive assumption to $\mathfrak{g}_1$, so that we can write $\mathfrak{g}_1 = \mathfrak{s} + Rad(\mathfrak{g}_1)$, with $\mathfrak{s}$ semisimple. Thus $\mathfrak{g} = \mathfrak{s} + (Rad(\mathfrak{g}_1) + Rad(\mathfrak{g}))$. Note that $Rad(\mathfrak{g}_1) + Rad(\mathfrak{g})$ is an ideal of $\mathfrak{g}$: Write an arbitrary element of $Rad(\mathfrak{g}_1) + Rad(\mathfrak{g})$ as $a + b$, where $a \in Rad(\mathfrak{g}_1)$ and $b \in Rad(\mathfrak{g})$. Write an arbitrary element of $\mathfrak{g}$ as $c + d$, where $c \in \mathfrak{g}_1$ and $d \in Rad(\mathfrak{g})$. Then

$$\begin{aligned}[a + b, c + d] &= [a, c + d] + [b, c + d] \\ &= [a, c] + [a, d] + [b, c + d],\end{aligned}$$

We have $[a, c] \in Rad(\mathfrak{g}_1)$ because $Rad(\mathfrak{g}_1)$ is an ideal of $\mathfrak{g}_1$, $[a, d], [b, c+d] \in Rad(\mathfrak{g})$ because $Rad(\mathfrak{g})$ is an ideal. Furthermore, $Rad(\mathfrak{g}_1) + Rad(\mathfrak{g})$ is solvable, because $(Rad(\mathfrak{g}_1) + Rad(\mathfrak{g}))^{(n)} \subseteq Rad(\mathfrak{g}_1)^{(n)} + Rad(\mathfrak{g})$, which follows from the fact that $Rad(\mathfrak{g}_1)$ is a sub-algebra and $Rad(\mathfrak{g})$ is an ideal of $\mathfrak{g}$. Hence, if $Rad(\mathfrak{g}_1)^{(n)} = 0$, and $Rad(\mathfrak{g})^{(m)} = 0$, then $(Rad(\mathfrak{g}_1) + Rad(\mathfrak{g}))^{(m+n)} = 0$. Hence, we have a solvable ideal of $\mathfrak{g}$ which contains $Rad(\mathfrak{g})$, so we conclude that our ideal is in fact $Rad(\mathfrak{g})$, yielding the equality

$$\mathfrak{g} = s + Rad(\mathfrak{g}).$$

What remains is to consider the case when $Rad(\mathfrak{g})$ is abelian. Consider the $\mathfrak{g}$-module structure $End\ \mathfrak{g}$ defined by:

$$a * m = (ad\ a)m - m(ad\ a)$$

for $a \in \mathfrak{g}$, $m \in End\ \mathfrak{g}$. Further, consider the following submodule of $End\ \mathfrak{g}$:

$$\tilde{M} = \{m \in End\ \mathfrak{g} | m(\mathfrak{g}) \subset Rad(\mathfrak{g}) \text{ and } m(Rad(\mathfrak{g})) = 0\}$$

Let $M = \tilde{M}/\tilde{R}$ where $\tilde{R} = \{ad(a) | a \in Rad(\mathfrak{g})\}$. Note that since $Rad(\mathfrak{g})$ is abelian, it acts trivially on M. Hence, M is actually an $\mathfrak{s}$-module. (where $\mathfrak{s} = \mathfrak{g}/Rad(\mathfrak{g})$)

Exercise 23.1: Show that $\tilde{M}$ is a submodule of $End(\mathfrak{g})$, and $\tilde{R}$ is a submodule of $\tilde{M}$.

Proof. $\tilde{M}$ is clearly a subspace of $End(\mathfrak{g})$. Given $a \in \mathfrak{g}$ and $m \in \tilde{M}$:

$$(a * m)(\mathfrak{g}) \subset (ad\ a)m(\mathfrak{g}) + m((ad\ a)\mathfrak{g}) \subset (ad\ a)Rad(\mathfrak{g}) + m(\mathfrak{g}) \subset Rad(\mathfrak{g})$$

$$(a * m)(Rad(\mathfrak{g})) \subset (ad\ a)m(Rad(\mathfrak{g})) + m(ad\ a)(Rad(\mathfrak{g})) \subset (ad\ a)0 + m(Rad(\mathfrak{g})) = 0$$

Hence $a * m \in \tilde{M}$. So $\tilde{M}$ is a submodule of $End(\mathfrak{g})$.

$\tilde{R}$ is also clearly a subspace of $End(\mathfrak{g})$. Given $ad\ a \in \tilde{R}$, with $a \in Rad(\mathfrak{g})$:

$$(ad\ a)(\mathfrak{g}) = [a, \mathfrak{g}] \subset Rad(\mathfrak{g})$$

$$(ad\ a)(Rad(\mathfrak{g})) = [a, Rad(\mathfrak{g})] = 0$$

Hence $\tilde{R}$ is a subspace of $\tilde{M}$. To see that it is a submodule, note that for $ad\ a \in \tilde{R}$ and $b \in \mathfrak{g}$:

$$b * (ad\ a) = (ad\ b)(ad\ a) - (ad\ a)(ad\ b) = [ad\ b, ad\ a] = ad([b, a]) \in \tilde{R}$$

□

Let P_0 be a projector of $\mathfrak{g}$ to Rad $\mathfrak{g}$, and consider the following 1-cocycle:

$$f : s \to M$$
$$a \to (\text{ad } \tilde{a})P_0 - P_0(\text{ad } \tilde{a}),$$

where $\tilde{a}$ is any preimage of a under the map $\mathfrak{g} \to \mathfrak{g}/\text{Rad } \mathfrak{g}$.

Exercise 23.2: The map f above is a well-defined 1-cocycle.

Proof. Let $b \in \text{Rad } \mathfrak{g}$.

$$(\text{ad } \tilde{a})P_0 - P_0(\text{ad } \tilde{a}) - ((\text{ad } \tilde{a} + \tilde{b})P_0 - P_0(\text{ad } \tilde{a} + \tilde{b})) =$$
$$(\text{ad } \tilde{b})P_0 - P_0(\text{ad } \tilde{b}) = \text{ad } \ -\tilde{b},$$

since we are assuming Rad $\mathfrak{g}$ is abelian, and since P_0 acts as the identity on Rad $\mathfrak{g}$. Hence, the map f is well-defined, since the target of our map is $\tilde{M}/\tilde{R}$.

It remains to make sure that our map is indeed a 1-cocycle. In the notation above, it's clear that $[\tilde{a}, \tilde{b}]$ maps to $[a, b]$. So, on one hand,

$$f([a, b]) = (\text{ad } [\tilde{a}, \tilde{b}])P_0 - P_0(\text{ad } [\tilde{a}, \tilde{b}])$$

and on the other hand

$$\begin{aligned} af(b) - bf(a) &= (\text{ad } a)f(b) - f(b)(\text{ad } a) - (\text{ad } b)f(a) + f(a)(\text{ad } b) \\ &= (\text{ad } a)((\text{ad } \tilde{b})P_0 - P_0(\text{ad } \tilde{b})) - ((\text{ad } \tilde{b})P_0 - P_0(\text{ad } \tilde{b}))(\text{ad } a) \\ &\quad - (\text{ad } b)((\text{ad } \tilde{a})P_0 - P_0(\text{ad } \tilde{a})) + ((\text{ad } \tilde{a})P_0 - P_0(\text{ad } \tilde{a}))(\text{ad } b) \\ &= (\text{ad } [\tilde{a}, \tilde{b}])P_0 - P_0(\text{ad } [\tilde{a}, \tilde{b}]). \end{aligned}$$

□

By the Fundamental Theorem on Cohomology, which may be applied because s is semi-simple, there exists $m \in M$ such that $a_m = (\text{ad } \widetilde{a})P_0 - P_0(\text{ad } \widetilde{a})$. But, simply writing out a_m and subtracting, this means that

$$(\text{ad } \widetilde{a})(P_0 - \widetilde{m}) - (P_0 - \widetilde{m})(\text{ad } \widetilde{a}) = \text{ad } r_a, \tag{1}$$

where $\widetilde{m}$ is a preimage of m in $\widetilde{M}$, and $r_a \in \text{Rad } \mathfrak{g}$.

Consider the projector $P = P_0 - \widetilde{m}$ of $\mathfrak{g}$ to Rad $\mathfrak{g}$.

Case 1: If all $r_a = 0$, then equation 1 above implies that ad $\tilde{a}$ commutes with P for all $a \in \mathfrak{g}$. In this case, let $s = \text{Ker } P$. Because P is a projector, $s \cap \text{Rad } \mathfrak{g} = 0$ and $\mathfrak{g} = s + \text{Rad } \mathfrak{g}$. So we get that $\mathfrak{g} = s \oplus \text{Rad } \mathfrak{g}$ as vector spaces. This is in fact a direct sum of ideals: that Rad $\mathfrak{g}$ is an ideal is clear, and s is an ideal because, by the commutativity mentioned above, for every $a \in \mathfrak{g}, s_1 \in s$,

$$0 = [a, P(s_1)] = P([a, s_1]).$$

Case 2: Now assume that not all $r_a = 0$. Let $\mathfrak{g}_1 = \{a \in \mathfrak{g} \mid P(\text{ad } a) = (\text{ad } a)P.\}$ This is a subalgebra by the first properties of the adjoint representation. It is a *proper* subalgebra by our assumption, and equation 1 together with the facts that $(\text{ad } r_a)P = 0$ since Rad $\mathfrak{g}$ is abelian and $P(\text{ad } r_a) = \text{ad } r_a$ since P is a projector, implies

$$P(\text{ad } a - r_a) = (\text{ad } a - r_a)P.$$

Hence, $\mathfrak{g} = \mathfrak{g}_1 + \text{Rad } \mathfrak{g}$.

Applying the inductive assumption, we can find a subalgebra s of $\mathfrak{g}_1$ such that $\mathfrak{g}_1 = s + \text{Rad } \mathfrak{g}_1$, a direct sum of vector spaces. Hence, $\mathfrak{g} = s + \text{Rad } \mathfrak{g} + \text{Rad } \mathfrak{g}_1$. As in the proof of Weyl's Theorem, this is equivalent to $\mathfrak{g} = s + \text{Rad } \mathfrak{g}$.

□

Proof. (**Mal'cev's Theorem**) Suppose the radical is abelian, and, from Levi's Theorem, we have $\mathfrak{g} = s \oplus \text{Rad } \mathfrak{g}$. Let P_s and P_r be the canonical projectors of $\mathfrak{g}$ onto s and Rad $\mathfrak{g}$, respectively. Let $a_r, b_r \in \text{Rad } \mathfrak{g}$, and $a_s, b_s \in s$. Say $a = a_r + a_s$, $b = b_r + b_s$. Such decompositions exist and are unique, because of our decomposition of $\mathfrak{g}$. Because Rad $\mathfrak{g}$ is an ideal, $P_s([a, b]) = [s_1, s_2]$, and thus P_s is a Lie algebra homomorphism. Furthermore, by again considering the decomposition $a = a_r + a_s$, it's clear that $P_r + P_s = 1$.

Let $s_1 \oplus \text{Rad } \mathfrak{g} = \mathfrak{g}$ be another Levi decomposition.

By the adjoint representation, since it is an ideal, Rad $\mathfrak{g}$ is an s_1 module. Define the 1-cocycle $f : s_1 \to \text{Rad } \mathfrak{g}$ by the formula $f(a) = P_r(a)$.

Exercise 23.3: Check that f is indeed a 1-cocycle.

Proof. Write decompositions of two elements $a, b \in s_1$ as above (in particular, $a_s \in s$, not s_1). Then

$$\begin{aligned} P_r([a_r + a_s, b_r + b_s]) &= [a_r, b_r] + [a_s, b_r] + [a_r, b_s] \\ &= [a_s, b_r] + [a_r, b_s] \end{aligned}$$

since we are assuming Rad $\mathfrak{g}$ to be abelian. This clearly equals

$$[a_r + a_s, P_r(b_r + b_s)] - [b_r + b_s, P_r(a_r + a_s)],$$

and so f is a 1-cocycle. □

By the Fundamental Theorem, this cocycle is trivial. So, there exists $r \in \text{Rad } \mathfrak{g}$ such that $P_r(a) = [a, r]$, for any $a \in s_1$. So we have that $P_r = -\text{ad } r$ on s_1.

Since Rad $\mathfrak{g}$ is abelian, $(\text{ad } r)^2 = 0$. Hence $\exp(\text{ad } r) = 1 + (\text{ad } r)$, and, from Exercise 9.2, that must be an automorphism of $\mathfrak{g}$. Call it σ.

Let $a \in s_1$. We have

$$\sigma(a) = (1 + \text{ad } r)a = (1 - P_r)a = P_s(a) \in s.$$

Hence, we have an automorphism σ of $\mathfrak{g}$ such that $\sigma(s_1) \subseteq s$. An automorphism is, of course, injective, and by the vector space decompositions $\mathfrak{g} = s \oplus \text{Rad } \mathfrak{g}$ and $\mathfrak{g} = s_1 \oplus \text{Rad } \mathfrak{g}$, we have that the dimensions of s and s_1 are equal. Hence, $\sigma(s_1) = s$.

Exercise 23.4: As in the proof of Levi's Theorem, reduce Mal'cev's Theorem in the case of non-abelian radical Rad $\mathfrak{g}$ to the case where Rad $\mathfrak{g}$ is abelian.

Proof. We take a slightly different route, following Jacobson, [Ja62]. Let $\mathfrak{N}$ denote the sum of all the nilpotent ideals of $\mathfrak{g}$. That $\mathfrak{N}$ is a nilpotent ideal is obvious. We will prove inductively that for every $i \geq 1$, there is an automorphism A_i of $\mathfrak{g}$ such that $A_i(s_1) \subseteq s + \mathfrak{N}^{(i)}$. Considering dimensions and the fact that $\mathfrak{N}$ is solvable, we will be able to conclude Mal'cev's Theorem.

Considering the equation

$$P_r([a_r + a_s, b_r + b_s]) = [a_r, b_r] + [a_s, b_r] + [a_r, b_s],$$

we see that $P_r([a, b]) \in [\mathfrak{g}, \text{Rad } \mathfrak{g}]$ for each $a, b \in s_1$.

Lemma 1. *With notation as above, we have that* $[\mathfrak{g}, \text{Rad } \mathfrak{g}] \subseteq \mathfrak{N}$.

Proof. We know from lecture five that $[\text{Rad } \mathfrak{g}, \text{Rad } \mathfrak{g}] \subseteq \mathfrak{N}$. By Engel's Theorem, this is equivalent to ad $[a_r, b_r]$ being nilpotent for $a_r, b_r \in \text{Rad } \mathfrak{g}$. Using the Jacobi identity, and the fact that Rad $\mathfrak{g}$ is an ideal, we get that ad $[a, b_r]$ is nilpotent for any $a \in \mathfrak{g}$. □

Lemma 2. *As s_1 is semi-simple, we have $[s_1, s_1] = s_1$.*

Proof. This follows from the structure theorem: the result holds for a simple Lie algebra t because it contain no non-trivial proper ideals, and thus $[t, t]$ of such an ideal is either 0 or the whole simple algebra, and as t is not abelian, it must be the latter. The result continues to hold under passage to direct sums. □

Now, consider any $a \in s_1$. We have just seen that we may write $a = [a_1, a_2]$, for some $a_1, a_2 \in s_1$. Hence, from what we proved above, $P_r(a) \in \mathfrak{N}$. We know that $P_r + P_s = 1$, so we have that $s_1 \subseteq s + \mathfrak{N}^{(1)}$. This provides the first step in our induction, with the identity automorphism playing the role of A_1.

Simplify notation by assuming that $s_1 \subseteq s + \mathfrak{N}^{(k)}$ (i.e., if necessary, replace s_1 with the isomorphic Lie Algebra $A_k(s_1)$). Because s is semi-simple, $s \cap \text{Rad } \mathfrak{g} = \emptyset$, and hence $P_r(s_1) \subseteq \mathfrak{N}^{(k)}$.

We can make $\mathfrak{N}^{(k)}$ into a s_1-module by defining $a \cdot z = [P_s(a), z]$. We check that this is indeed a module:

$$\begin{aligned}
[a, b] \cdot z &= [P_s([a, b]), z] \\
&= [[P_s(a), P_s(b)], z] \\
&= -[[P_s(b), z], P_s(a)] - [[z, P_s(a)], P_s(b)] \\
&= [P_s(a), [P_s(b), z]] + [P_s(b), [z, P_s(a)]] \\
&= [P_s(a), [P_s(b), z]] - [P_s(b), [P_s(a), z]] \\
&= a \cdot (b \cdot z) - b \cdot (a \cdot z),
\end{aligned}$$

as required. As $\mathfrak{N}^{(k+1)}$ is an ideal, we can consider also $\mathfrak{N}^{(k)}/\mathfrak{N}^{(k+1)}$ as an s_1-module, with action $a \cdot \overline{z} = \overline{[P_s(a), z]}$.

From what we saw two paragraphs ago, $[P_r(a), P_r(b)] \in \mathfrak{N}^{(k+1)}$, for $a, b \in s_1$. So,

$$\overline{P_r([a, b])} = \overline{[a_s, b_r]} - \overline{[b_s, a_r]} = a \cdot \overline{b_r} - b \cdot \overline{a_r}.$$

Define the map $f : a \to \overline{P_r(a)}$. The map is linear because the projector is linear, and the above equation says precisely that f is a one-cocycle. Hence, by the fundamental theorem on cohomology, there is some element $\overline{z} \in \mathfrak{N}^{(k)}/\mathfrak{N}^{(k+1)}$ such that, for any $a \in s_1$, we have $\overline{P_r(a)} = a \cdot \overline{z} = \overline{[P_r(a), z]}$, where z is any lift of $\overline{z}$.

Let $A = \exp(\text{ad } z)$. Again, from exercise 9.2, this is an automorphism. We have

$$\begin{aligned} A(a) &= a + [z, a] + \frac{1}{2!}[z, [z, a]] + \cdots \\ &\equiv a + [z, a] \bmod \mathfrak{N}^{(k+1)} \\ &\equiv a_r + a_s + [z, a_r] + [z, a_s] \bmod \mathfrak{N}^{(k+1)} \end{aligned}$$

and we know $a_s \in \mathfrak{N}^{(k)}$ and $\overline{a_r} = \overline{[a_r, z]}$, so

$$\equiv a_s \bmod \mathfrak{N}^{(k+1)}.$$

Thus, we have that $A(s_1) \subseteq s + \mathfrak{N}^{(k+1)}$, and we proceed by induction. □

As already mentioned, the proof is now complete, when we consider that the dimensions of s and s_1 must be the same. □

References

[Ja62] N. Jacobson, *Lie Algebras*, Interscience Publishers, 1962.