18.745: LECTURE 25

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Let $\mathfrak{g}$ be as in the last lecture - finite dimesional semisimple lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra, and $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset \Delta_+ \subset \Delta$, as before, a system of simple roots. We have the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$, $\mathfrak{b} = \mathfrak{h}_+ + \mathfrak{n}_+$, with $\mathfrak{b}$ - a Borel subalgebra, and $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}_+$. Let $(\cdot, \cdot)$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, let $\rho = \frac{1}{2}\Sigma_{\alpha\in\Delta_+}\alpha$. Let $\{E_i, H_i, F_i\}$ be the Chevalley generators satisfying $H_i = \frac{2\nu^{-1}(\alpha_i)}{(\alpha_i,\alpha_i)}$, $E_i \in \mathfrak{g}_\alpha$, $F_i \in \mathfrak{g}_{-\alpha}$ and such that $< E_i, H_i, F_i >$ form the standard basis of $\mathrm{sl}_2(\mathbb{F})$. Recall that E_i's (respectively F_i's) generate $\mathfrak{n}_+$ (respectively $\mathfrak{n}_-$) and H_i's form a basis of $\mathfrak{h}$. We have the weight lattice $P = \{\lambda \in \mathfrak{h}^* | \lambda(H_i) \in \mathrm{Z} \text{ for all } i = 1, \dots r\}$. Note that $Q \subset P$, since $\alpha_i(H_j) \in \mathrm{Z}$. Define the subset $P_+ = \{\lambda \in \mathfrak{h}^* | \lambda(H_i) \in \mathrm{Z}_+ \text{ for all } i = 1, \dots r\}$, called the set of dominant integral weights.

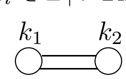
Theorem 25.1. *(Cartan) The $\mathfrak{g}$-modules $\{L(\Lambda)\}_{\Lambda\in P_+}$ are, up to isomorphism, all irreducible finite-dimensional $\mathfrak{g}$-modules. (Recall from previous lectures that $L(\Lambda)$ is the irreducible heigest weight module with heighest weight λ.)*

Theorem 25.2. *(Weyl's dimension formula)* $\dim L(\Lambda) = \Pi_{\alpha\in\Delta}\frac{(\Lambda+\rho,\alpha)}{(\rho,\alpha)}$ *provided that* $\Lambda \in P_+$

Note. $\rho(H_i) = 1$ for all i. Indeed, $s_i(\rho) = s_i\left(\frac{1}{2}\alpha_i + \Sigma_{\alpha\in\Delta_+\setminus\{\alpha_i\}}\alpha\right) = -\frac{1}{2}\alpha_i + \Sigma_{\alpha\in\Delta_+\setminus\{\alpha_i\}}\alpha = \rho - \alpha_i$. But for any λ we have $s_i(\lambda) = \lambda - \lambda(H_i)\alpha_i$. Hence $\rho(H_i) = 1$

Example 25.3. $\mathfrak{g} = \mathrm{sl}_2$. All finite dimensional irreducible sl-modules are $L(\Lambda(m\rho))$ for $m \in \mathrm{Z}_+$ (observe that $m\rho(H) = m$). $\dim L(m\rho) = m + 1$.

Example 25.4. $\mathfrak{g} = \mathrm{sl}_3$. We have $\Pi = \{\alpha_1, \alpha_2\}$, $(\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\rho = \alpha_1 + \alpha_2$, $\rho(\alpha_i) = 1$, $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$, where $(\Lambda_i, \alpha_j) = \delta_{ij}$. By Cartan's theorem, $\dim L(\Lambda) < \infty$ iff $k_1, k_2 \in \mathrm{Z}_+$. We compute $(\Lambda + \rho, \alpha_1) = k_1 + 1$, $(\Lambda + \rho, \alpha_2) = k_2 + 1$, and $(\Lambda + \rho, \alpha_1 + \alpha_2) = k_1 + k_2 + 1$, so the Weyl's dimension formula gives $\dim L(\Lambda) = \frac{(k_1+1)(k_2+1)(k_1+k_2+1)}{2}$.

In general, we may write $\Lambda = \Sigma_i k_i \Lambda_i$, where $\Lambda_i(H_j) = \delta_{ij}$. Then $\dim L(\Lambda) < \infty$ iff $k_i \in \mathrm{Z}_+$. These k_i are called *labels* of the heighest weight. They are depicted on the Dynkin diagram: 

Operations on modules can then often be described by manipulations of such labeled Dynkin diagrams.

Proof of Theorem 25.1. First suppose that V is a finite dimensional irreducible $\mathfrak{g}$-module. By Lie's theorem, since $\mathfrak{b}$ is solvable, there exists a non-zero vector $v \in V$ and $\lambda \in \mathfrak{b}^*$ such that $b(v) = \lambda(b)v$ for any $b \in \mathfrak{b}$. Now if $n \in \mathfrak{n}_+ = [\mathfrak{b}, \mathfrak{b}]$, then $\lambda(n) = \Sigma_i\lambda([b_i, b_i']) = \Sigma_i\lambda(b_i)\lambda(b_i') - \lambda(b_i')\lambda(b_i) = 0$. So we have $hv = \lambda(h)v$ and $\mathfrak{n}_+v = 0$. Also $\mathcal{U}(\mathfrak{g})v = V$, since V is irreducible, and $\mathcal{U}(\mathfrak{g})v$ is a non-zero submodule (it contains v). Hence $V = L(\lambda)$. Why is $\lambda \in P_+$? This is because V is a finite-dimeensional module with respect to $< E_i, H_i, F_i > \cong \mathrm{sl}_2$, and since $E_iv = 0$, $H_iv = \lambda(H_i)v$, by the key lemma on sl_2-modules we conclude that $\lambda(H_i) \in \mathrm{Z}_+$. It remains to show that $\dim L(\Lambda) < \infty$ if $\Lambda \in P_+$. This follows from the Weyl's dimension formula, which we will in turn deduce from the Weyl's character formula. □

Definition. Let V be a $\mathfrak{g}$-module, such that $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$, where $V_\mu = \{v | hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}$ (we assume $\dim V < \infty$). Then the (formal) character of V is $\mathrm{ch}V = \Sigma_{\mu\in\mathfrak{h}^*}(\dim V_\mu)e^\mu$, where e^μ are formal symbols, which obey the property of exponentials, $e^\lambda e^\mu = e^{\lambda+\mu}$, $e^0 = 1$.

Theorem 25.5. *(Weyl's character formula.) Let $R = \Pi_{\alpha\in\Delta_+}(1-e^{-\alpha})$. Provided that $\Lambda \in P_+$, one has:* $e^\rho R\ \mathrm{ch}L(\Lambda) = \Sigma_{w\in W}(\det w)e^{w(\Lambda+\rho)}$

Example 25.6. sl_2. We get $e^{\frac{\alpha}{2}}(1-e^{-\alpha})\ \mathrm{ch}L(m\rho) = e^{(m+1)\rho} - e^{-(m+1)\rho}$, so that $\mathrm{ch}L(m\rho) = \frac{e^{(m+1)\rho}-e^{-(m+1)\rho}}{e^\rho - e^{-\rho}} = e^{m\rho} + e^{(m-2)\rho} + \ldots + e^{-m\rho}$, which exactly corresponds to what we would expect from last lecture, since applying F means subtracting $\alpha = 2\rho$.

Derivation of Theorem 25.2 from Theorem 25.5. Given $v \in \mathfrak{h}^*$, consider the linear map F_ν characterised by $e^\lambda \mapsto e^{t(\nu,\lambda)}$. It maps linear combinations to linear combinations and products to products. Applying F_ρ to both sides of Weyl's character formula:

$$e^{t(\rho,\rho)}\Pi_{\alpha\in\Delta_+}\left(1-e^{-t(\rho,\alpha)}\right)\Sigma_\lambda\left(\ \dim L(\Lambda)_\lambda\right)e^{t(\rho,\lambda)} = \Sigma_{w\in W}(\det w)e^{t(\rho,w(\Lambda+\rho))} =$$

$$= \Sigma_{w\in W}(\det w)e^{t(w^{-1}(\rho),\Lambda+\rho)} = \Sigma_{w\in W}(\det w)e^{t(w(\rho),\Lambda+\rho)} = F_{\Lambda+\rho}\left(\Sigma_{w\in W}(\det w)e^{w(\rho)}\right).$$

Letting $\Lambda = 0$ in the Weyl character formula, we get the Weyl denominator identity:

$$e^\rho R\cdot 1 = \Sigma_{w\in W}\ (\det w)\ e^{w(\rho)}.$$

Substituting in, we obtain

$$F_{\Lambda+\rho}\left(\Sigma_{w\in W}(\det w)e^{w(\rho)}\right) = e^{t(\Lambda+\rho,\rho)}\Pi_{\alpha\in\Delta_+}\left(1-e^{-t(\Lambda+\rho,\alpha)}\right).$$

Together with the first eguality, this gives

$$e^{t(\rho,\rho)}\Sigma\ \dim L(\Lambda)_\lambda e^{-t(\rho,\alpha)} = e^{t(\Lambda+\rho,\rho)}\Pi_{\alpha\in\Delta_+}\frac{1-e^{-t(\Lambda+\rho,\alpha)}}{1-e^{-t(\rho,\alpha)}}$$

Taking the limit as t goes to 0, we get, by L'Hopital's rule, $\dim L(\Lambda) = \lim_{t\to 0}\Pi_{\alpha\in\Delta_+}\frac{(\Lambda+\rho,\alpha)e^{-t(\Lambda+\rho,\alpha)}}{(\rho,\alpha)e^{-t(\rho,\alpha)}} = \Pi_{\alpha\in\Delta_+}\frac{(\Lambda+\rho,\alpha)}{(\rho,\alpha)}$ □

Proof of the Weyl's character formula. The Weyl group acts on the linear combinations of formal exponentials in an obvious way: $w\left(\Sigma_\lambda c_\lambda e^\lambda\right) = \Sigma_\lambda c_\lambda e^{w(\lambda)}$.

Lemma 1. If $\Lambda(H_i) \in \mathrm{Z}_+$, then $\mathrm{ch}L(\Lambda)$ is r_i-invariant.

Proof. By the key sl_2 lemma, $F_i^{\Lambda(H_i+1)}v_\Lambda$ is a singular vector of $L(\Lambda)$ (it is killed by E_i by the key lemma, and by E_j for $j \neq i$ since F_i and E_j commute). As $L(\Lambda)$ is irreducible, we conclude that $F_i^{\Lambda(H_i)+1}v_\Lambda = 0$. But $L(\Lambda) = \mathcal{U}(\mathfrak{g})v_\Lambda$. Since $(\mathrm{ad}F_i)^N u = 0$ for all $N \gg 0$, given $u \in \mathcal{U}(\mathfrak{g})$, we conclude that $F_i^N v = 0$ for $N \gg 0$, given $v \in V$. It is easy to deduce, using Weyl's complete reducibility theorem, that $L(\Lambda)$ is isomorphic to a direct sum of irreducible $\mathrm{sl}_2 =< E_i, H_i, F_i >$-modules, say $V_j : L(\Lambda) = \oplus_j V_j$. But for each V_j the lemma holds, since $V_j \cong \mathrm{sl}_2$-module $L(m\rho)$. Hence the lemma holds for $L(\Lambda)$ as well. □

Lemma 2. For the Verma module $M(\Lambda)$ we have $R\ \mathrm{ch}M(\Lambda) = e^\lambda$

Proof. By Proposition 2 from the last lecture, vectors $E_{-\beta_1}^{k_1}\ldots E_{-\beta_N}^{k_N}$ form a basis of $M(\Lambda)$. Hence $\mathrm{ch}M(\Lambda) = \Sigma_{(k_1,\ldots k_N)\in\mathrm{Z}_+^N}e^{\Lambda-k_1\beta_1\ldots-k_N\beta_N} = e^\Lambda\Pi_{\beta\in\Delta_+}(1+e^{-\beta}+e^{-2\beta}+\ldots)$. Multiplying both sides by R we get the desired result. □

Lemma 3. $w(e^\rho R) = (\det w)e^\rho R$

Proof. Since s_i's generate W, it suffices to prove $s_i(e^\rho R) = -e^\rho R$. Indeed, $s_i(e^\rho R) = s_i(e^\rho(1-e^{-\alpha_i})\left(\Pi_{\alpha\in\Delta_+\setminus\alpha_i}(1-e^{-\alpha})\right) = e^{\rho-\alpha_i}(1-e^{\alpha_i})\Pi_{\alpha\in\Delta_+\setminus\alpha_i}(1-e^{-\alpha}) = -e^\rho R$, as wanted. □

Lemma 4. Let $\Lambda \in \mathfrak{h}_{\mathbb{R}}^*$ and let V be a heighest weight module with heighest weight Λ. Let $D(\Lambda) = \{\Lambda - \Sigma k_i\alpha_i, k_i \in \mathrm{Z}_+\}$. Then $\mathrm{ch}V = \Sigma_{\lambda\in B(\Lambda)}a_\lambda\ \mathrm{ch}L(\lambda)$, where $a_\Lambda = 1$, $a_\lambda \in \mathrm{Z}_+$, and $B(\Lambda) = \{\lambda \in D(\Lambda) | (\Lambda+\rho,\Lambda+\rho) = (\lambda+\rho,\lambda+\rho)\}$.

Proof. By induction on $\dim V = \Sigma_{\lambda \in B(\Lambda)} \dim V_\lambda < \infty$, since $B(\Lambda)$ is finite (by Proposition 1 from last lecture). If v_λ is the only singular vector of V, then $V = L(\Lambda)$ and we are done. If we have another singular vector v_λ then by Proposition 1 from last time $\lambda \in B(\Lambda)$. Let $U = \mathcal{U}(\mathfrak{g})v_\lambda$, a heighest weight submodule of V. Then we have an exact sequence of $\mathfrak{g}$-modules $0 \to U \to V \to V/U \to 0$. Then $\mathrm{ch}V = \mathrm{ch}U + \mathrm{ch}U/V$, and we apply the induction assumption to both summands. □

Lemma 5. In the assumptions of Lemma 4 and presuming V is irreducible, we have $\mathrm{ch}V = \Sigma_{\lambda \in B(\lambda)} b_\lambda \mathrm{ch}M_\lambda$, where $b_\Lambda = 1$ and $b_\lambda \in Z$.

Proof. By Lemma 4 we have for any $\mu \in M(\Lambda)$: $\mathrm{ch}M_\mu = \Sigma_{\lambda \in B(\mu)} a_{\lambda,\mu} \mathrm{ch}L(\lambda)$. Now $B(\Lambda) = \{\lambda_1, \ldots, \lambda_s\}$, where $\lambda_i - \lambda_j \notin \{\Sigma k_i \alpha_i | k_i \in \mathrm{Z}_+\}$ if $i > j$. We therefore have a system of linear equations $\mathrm{ch}M_\mu = \Sigma_{\lambda \in B(\mu)} a_{\lambda,\mu} \mathrm{ch}L(\lambda)$, for which the matrix $(a_{ij})_{ij}$ is upper triangular matrix of integers with ones on the diagonal, and so its inverse, which expresses $\mathrm{ch}L(\Lambda)$'s in terms of $\mathrm{ch}M(\mu)$'s for $\mu \in B(\Lambda)$ is a matrix of integers with ones on the diagonal as well, and we are done. □

Now, to deduce the theorem from the lemmas, observe that by Lemma 5 $\mathrm{ch}V = \Sigma_{\lambda \in B(\lambda)} a_\lambda \mathrm{ch}M_\lambda$, where $a_\Lambda = 1$ and $a_\lambda \in Z$ We multiply both sides by $e^\rho R$ and use Lemma 2 to obtain $e^\rho R \ \mathrm{ch}L(\Lambda) = \Sigma_{\lambda \in B(\lambda)} a_\lambda e^{\rho+\lambda}$, $a_\Lambda = 1$, $a_\lambda \in Z$. By Lemma 1 $\mathrm{ch}L(\Lambda)$ is W-invariant, and by Lemma 3 $e^\rho R$ is W-anti-invariant (i.e. multiplied by the determinant). Hence the left hand side of the equation is anti-invariant, and therefore so is the right hand side. We have

$$e^\rho R \ \mathrm{ch}L(\Lambda) = \Sigma_{w \in W} (\det w) e^{w(\Lambda+\rho)} + \Sigma_{\lambda \in B(\Lambda) \setminus \{\Lambda\}, \lambda+\rho \in P_+} a_\lambda \Sigma_{w \in W} (\det w) e^{w(\lambda+\rho)}$$

It remains to show that the second term in this sum is empty, i.e. that there are no λ with $\lambda + \rho \in P_+$, $\lambda = \Lambda - \alpha$ for $\alpha = \Sigma k_i \alpha_i, k_i \in \mathrm{Z}_+, \alpha \neq 0$ and $(\lambda + \rho, \lambda + \rho) = (\Lambda + \rho, \Lambda + \rho)$. Indeed, for such a λ we would have $0 = (\Lambda + \rho, \Lambda + \rho) - (\lambda + \rho, \lambda + \rho) = (\lambda + \Lambda + 2\rho, \alpha) > 0$, since $(\Lambda, \alpha_i) = \frac{2\Lambda(H_i)}{(\alpha_i, \alpha_i)} \geq 0$ and similarly $(\lambda + \rho, \alpha_i) \geq 0$, and $(\rho, \alpha) > 0$ since $\frac{2}{(\alpha_i, \alpha_i)} > 0$. This gives a contradictionand complets the proof.

□