18.745: LECTURE 3

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Exercise 1. *Show that the complete list of n-dimensional $\mathfrak{g}$ with $\dim \mathrm{Z}(\mathfrak{g}) = n-2$ is*

(1) $H_1 \oplus Ab_{n-3}$, with H_1 as in Homework 2.
(2) $b \oplus Ab_{n-2}$, with b and Ab as in Homework 2.

Proof. Let $B = \{x_1, \ldots, x_{n-2}\}$ be a basis for $\mathrm{Z}(\mathfrak{g})$. Let $a, b \in \mathfrak{g}$ be some vectors such that $\mathrm{span}(B \cup \{a, b\}) = \mathfrak{g}$. a, b are linearly independent since they span a 2-dimensional subspace. The commutator of b is 0 on an $n-1$ dimensional subspace of $\mathfrak{g}$: $[b, x_i] = 0, [b, b] = 0$, so we must have $[a, b] \neq 0$, since $b \notin \mathrm{Z}(\mathfrak{g})$. Let $c = [a, b]$; we then have two cases:

(1) If $c \in \mathrm{Z}(\mathfrak{g})$, we call $a = p_1$ and $b = q_1$, and we see that $\{a, b, c\}$ is just H_1.
(2) Otherwise, write c in the basis $B \cup \{a, b\}$: $c = \sum z_i x_i + z_a a + z_b b$. We know that at least one of $z_a, z_b$is non-zero. Without loss of generality, let this be z_a. Then, $[c, b] = z_a[a, b] = z_a c$ (since $\mathrm{ad}\, b$ kills the other basis vectors). Renormalizing $b := z_a^{-1} b$, we have $[c, b] = c$, so b, c span the non-commutative subalgebra of dimension 2. Observe that (since $z_a \neq 0$), $B \cup \{b, c\}$ is a basis for $\mathfrak{g}$, so we have the desired result.

□

1. Homework policy

The exercises from the previous two lectures are due every Tuesday. Lecture write-ups should be handed in within two weeks of the date of the lecture.

2. Representations

Representations are very useful tools for analysing the structure of Lie algebras. There are some theorems that only have proofs using representation theory, although they can be stated without it.

Definition 2.1. A *representation* of a Lie algebra $\mathfrak{g}$ in a vector space V over a field $\mathbb{F}$ is a homomorphism $\pi : \mathfrak{g} \to \mathrm{gl}\, V$. In particular, this means that $\pi([a, b]) = [\pi(a), \pi(b)]$.

3. Examples

Example 3.1. $\pi(a) = 0$ for all $a \in \mathfrak{g}$. This is called the *trivial representation.*

Example 3.2. Let $\mathfrak{g}$ be a subalgebra of $\mathrm{gl}\, V$, and let $\pi(a) = a$. This is called the defining (tautological, identity) representation.

Example 3.3. The adjoint representation, $\mathrm{ad} : \mathfrak{g} \to \mathrm{gl}\, \mathfrak{g}$ is defined by $\mathrm{ad}(a) \to (\mathrm{ad}\, a)$.

To check that $\mathrm{ad}[a, b] = [\mathrm{ad}\, a, \mathrm{ad}\, b]$, we apply it to c, getting

$$[[a, b], c] \stackrel{?}{=} (\mathrm{ad}\, a\, \mathrm{ad}\, b - \mathrm{ad}\, b\, \mathrm{ad}\, a)c = [a, [b, c]] - [b, [a, c]],$$

which is just the Jacobi identity. So, this linear map is indeed a representation.

Note. In these lecture notes, I will denote the center of $\mathfrak{g}$ as $\mathrm{Z}(\mathfrak{g})$.

Corollary 3.4. ad *defines an embedding of* $\mathfrak{g}/\mathrm{Z}(\mathfrak{g})$ *into* $\mathrm{gl}\, \mathfrak{g}$*, since* $\mathrm{Ker}\, \mathrm{ad} = \mathrm{Z}(\mathfrak{g})$.

In particular, if $\mathrm{Z}(g) = 0$, we embed $\mathfrak{g}$ as a subalgebra of $\mathrm{gl}\, \mathfrak{g}$. It's thus natural to ask: can always embed $\mathfrak{g}$ in some $\mathrm{gl}\, V$? The answer is (stated without proof):

Theorem 3.5. *(Ado's Theorem) In fact, any finite-dimensional Lie algebra can be embedded in some* $\mathrm{gl}\, V$*,* V *finite-dimensional.*

There are some general ways of constructing representations.

(1) Let $\pi : \mathfrak{g} \to \mathrm{gl}\, V$ be a representation, and there $U \subset V$ is an invariant subspace, i.e.$\pi(\mathfrak{g})U \subset U$. Then, the restriction $\pi|_U : U \to \mathrm{gl}\, U$ is a representation.
(2) Taking U as above, we can look at the quotient space V/U as well. Then, $\pi|_{V/U} : \mathfrak{g} \to \mathrm{gl}(V/U)$. This warrants some explanation. π takes $g \in \mathfrak{g}$ to $A \in \mathrm{End}\, V$; U is invariant, so $A \subset AU$. Thus, it can be viewed as an operator on V/U: $A(v+U) = Av + U$.
(3) Take 2 representation π_1, π_2 of $\mathfrak{g}$ in vector spaces V_1 and V_2, respectively. Then, we can consider the direct sum: for $g \in \mathfrak{g}$, $(\pi_1 \oplus \pi_2)(g) = \pi_1(g) \oplus \pi_2(g) \in V_1 \oplus V_2$.

4. Engel's Theorem

Now we prove the first non-trivial theorem about Lie algebras. First, a reminder:

Definition. An operator A on a vector space is nilpotent if there exists $n > 0$ such that $A^n = 0$.

Theorem. *(Engel's Theorem) Let $\mathfrak{g} \subset \mathrm{gl}\, V$ be a subalgebra consisting of nilpotent operators. Assume V is finite-dimensional. Then, there exists a $v \neq 0$ in V such that every $g \in \mathfrak{g}$ annihilates v.*

Proof. We will proceed by induction on the dimension of $\mathfrak{g}$ (which is a subspace of the finite-dimensional $\mathrm{End}\, V$).

The base case is $\dim V = 1$. Then, $\mathfrak{g} = \mathbb{F}a$, with a is a nilpotent operator on V. Take N such that $a^N = 0$, but $a^{N-1} \neq 0$. Then, $\exists v$ such that $a^{N-1}v \neq 0$. However, $a(a^{N-1}v) = 0$, so $v' = a^{N-1}v$ is the desired vector.

Assume that the theorem holds for all $\mathfrak{g}$ of dimension $< n$. We will prove that it holds for $\mathfrak{g}$ of dimension $n \geq 2$.

Take a proper subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of the maximal possible dimension. Since $\mathbb{F}a$ is a subalgebra of $\mathfrak{g}$. $\dim \mathfrak{h} \geq 1$. Consider a representation of $\mathfrak{h}$ given by $\pi : h \to \mathrm{ad}\, h, h \in \mathfrak{h}$, π acting on $\mathfrak{g}$. The fact that $\mathfrak{h}$ is a subalgebra means that $\mathfrak{h}$ is an invariant subspace with respect to π.

Take $\pi' = \mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}$ to be the representation of $\mathfrak{h}$ in $\mathfrak{g}/\mathfrak{h}$. In Lecture 2, we proved a lemma: if $g \in \mathfrak{g}$ is nilpotent, then so is $\mathrm{ad}\, g$. Therefore, $\mathrm{ad}\, h$, considered as a Lie algebra over $\mathfrak{g}/\mathfrak{h}$, consists of nilpotent operators. We know that $\dim \mathfrak{h} < \dim \mathfrak{g}$, so by induction, there is a non-zero vector $\bar{a} \in V$ that's annihilated by all $\pi(h), h \in \mathfrak{h}$. Let a be some preimage of $\bar{a}$ in $\mathfrak{g}$ under the map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Then, we have $[\mathfrak{h}, a] \subset \mathfrak{h}$; in other words, $\mathfrak{h} \oplus \mathbb{F}a$ is a subalgebra of $\mathfrak{g}$. It isn't equal to $\mathfrak{h}$, and so, by maximality of $\mathfrak{h}$ it must be $\mathfrak{g}$. Hence, $\mathfrak{h}$ must have codimension 1.

Applying the inductive assumption to $\mathfrak{h}$, we get a nonzero $v \in V$ such that $\mathfrak{h}v = 0$. Thus, the subspace $U \subset V$ of all vectors annihilated by $\mathfrak{h}$ is non-empty.

U is a-invariant. Take $u \in U$, then we have this equivalence, $a(u) \in U \Leftrightarrow h(a(u)) = 0 \forall h \in \mathfrak{h}$. Expanding the second formulation, we have $ha(u) = ([h,a] + ah)u$. $[h,a] \in \mathfrak{h}$, so the first term is zero; similarly $h(u) = 0$, so $a(h(u)) = 0$. Hence, U is indeed a-invariant. Thus, we can consider the restriction of a to U. a is nilpotent, so $\exists u \in U$ such that $au = 0$, but $u \neq 0$. This u is the desired vector. □

It's natural to ask: What if $\mathfrak{g}$ is a subspace, but not a subalgebra? Does the theorem still hold? It turns out that the answer is no. The following exercise provides a counterexample.

Exercise 2. *Find a 2-dimensional subspace S of 3×3 matrices that consists of nilpotent matrices, so that no $0 \neq v \in \mathbb{F}^3$ is annihilated by every $s \in S$.*

We want two matrices A, B such that any linear combination has characteristic polynomial x^3, but with non-overlapping eigenspaces. If we take

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

it has a 0th eigenspace, which is spanned by $e_1 = (1,0,0)$. Then, B mustn't annihilate e_1, but needs to kill some linear combination of e_2 and e_3. Suppose $Be_3 = 0$. Then,

$$B = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 0 \end{pmatrix}, (B - \lambda I) = \begin{pmatrix} a-\lambda & b+g & 0 \\ c & d-\lambda & g \\ e & f & -\lambda \end{pmatrix}$$

We want

$$\begin{aligned} -\lambda^3 &= (a-\lambda)(-\lambda(d-\lambda) - fg) - (b+g)(-\lambda c - eg) = \\ &= g(be - af + eg) + (g(f+c) + bc - ad)\lambda + (a+d)\lambda^2 - \lambda^3 \end{aligned}$$

no matter what g is. Thus, $e = 0$, $f = -c$,$af = 0$, $a = -d$, and $ad = bc$. Assume $f \neq 0$; then, $a = d = 0$, and $b = 0$. We can thus take

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

We see that its 0th eigenspace is spanned by $(0,0,1)$, and so no $v \in \mathbb{F}$is killed by both A and B, as we wanted. By construction, the subspace spanned by A and B consists of nilpotent matrices, so we have the desired result.

5. Equivalent Formulations

In this section, we present some corollaries of Engel's Theorem.

(1) Let $\pi :\to \mathrm{gl}\, V$ be a representation of $\mathfrak{g}$ in a finite dimensional vector space V, such that all $\pi(g), g \in \mathfrak{g}$ are nilpotent operators. Then, there exists nonzero $v \in V$ that gets annihilated by all $\pi(g)$. Moreover, there exists a basis of V, in which all $\pi(g)$ are strictly upper-triangular.

Proof. $\pi(\mathfrak{g})$ is a subalgebra of $\mathrm{gl}\, V$; applying Engel's theorem, we find a nonzero $v \in V$ such that $\pi(g)v = 0 \forall g \in \mathfrak{g}$.

To construct the basis, take a $v_1 = v$ as above. Then, $\mathbb{F}v_1$ is $\pi(g)$-invariant, since it gets annihilated. (Hence, every $\pi(g)$ has a column of zero coefficients corresponding to v_1 if it's used as a basis vector). By invariance of $\mathbb{F}v_1$, we may consider $V_2 = V/\mathbb{F}v_1$, and the corresponding representation π_1 of $\mathfrak{g}$ induced by π. The operators $\pi_1(g)$ are also nilpotent; we apply Engel's Theorem again to get v_2' that's annihilated by all $\pi_1(g)$. Then, a pre-image $v_2 \in V$ of v_2' is sent to $\mathbb{F}v_1$by every $\pi(g)$. We proceed to make V_3 and get v_3, etc. In the end, we have the desired basis. □

(2) A subalgebra $\mathfrak{g} \subset \mathrm{gl}\, V, \dim V < \infty$ consists of nilpotent matrices iff in some basis of V, all matrices of $g \in \mathfrak{g}$ are strictly upper-triangular.

Proof. If all matrices of $\mathfrak{g}$ are strictly upper-triangular, they are all nilpotent. To see this, take $e_1, \dots e_n$ to be the standard basis vectors. Then (with c standing for arbitrary constants), $Ae_1 = 0, Ae_2 = ce_1 \Rightarrow A^2e_2 = 0, Ae_3 = ce_1 + ce_2 \Rightarrow A^3e_3 = 0, \dots$. Hence, $A^n e_i = 0$ for all i.

The converse follows entirely from 1, by applying it with π chosen to be the defining representation. □

Here is another exercise to conclude the lecture.

Exercise 3. *Compute the centers of the Lie algebras* $gl_n(\mathbb{F}), sl_n(\mathbb{F}), so_n(\mathbb{F}), sp_n(\mathbb{F})$.

(1) $gl_n(\mathbb{F})$ is the entire space of matrices. If $n = 1$, the entire space is commutative (and hence $\mathrm{gl}_1(\mathbb{F})$ is its own center). Otherwise, denote by $M_{i,j}$ the matrix of zeros with a single 1 at position i, j. Then, if $x \in \mathrm{Z}(gl_n(\mathbb{F}))$, $M_{i,j}x = xM_{ij}$. The first term is the jth row of x placed as the ith row in the product. The second term is the ith column of x placed as the jth column in the product. The other entries in both matrices are 0. Therefore, everything in the jth row of x (except for j, j) must be equal to 0. Hence, x must be diagonal. Moreover,

notice that $(M_{i,j}x)_{i,j} = x_{j,j}$, while $(xM_{i,j})_{i,j} = x_{i,i}$. So, $x_{i,i} = x_{j,j}$. This forces $x = aI$ for some $a \in \mathbb{F}$.

(2) $sl_n(\mathbb{F})$ is the algebra of matrices with trace 0. If $n = 1$, the algebra and its center are trivial. When $n \geq 2$, the matrices $M_{i,j}, i \neq j$ used before are in the algebra. Hence, the same argument shows that $\mathrm{Z}(sl_n(\mathbb{F})) \in \{aI | a \in \mathbb{F}\}$. But, since these are trace 0 matrices, $a = 0$, and $\mathrm{Z}(sl_n(\mathbb{F})) = 0$.

(3) We have a symmetric B. Suppose $x \in \mathrm{Z}(so_n(\mathbb{F}))$; then, we have $x^T B + Bx = 0$ and $xz = zx$ for any $z \in \mathfrak{g}$. Let $y = (M_{ij} - M_{ji})$, and $z = B^{-1}y$, then, it is in $\mathfrak{g}$:

$$z^T B + Bz = y^T (B^{-1})^T B^T + BB^{-1}y = y^T + y = 0$$

Multiplying the commutation relation by B, we have $0 = Bxz - Bzx = -x^T Bz - Bzx = xy + yx$. Since y is antisymmetric, this is equivalent to $(yx)^T = (yx)$. Observe that yx is all zeros, except for a copy of jth row of x in the ith row place, and a negated copy of the ith row of x in the jth row place. The fact that yx is symmetric means that all but four positions of x are forced to be 0. We get no constraint on $x_{i,j}$ and $x_{j,i}$, since these land on the diagonal in xy. However, we do get that $x_{i,i} = -x_{j,j}$. Supposing that $n \geq 3$, this means we can zero out all the off-diagonal elements. Additionally, we would get $x_{i_1,i_1} = -x_{i_2,i_2}, x_{i_2,i_2} = -x_{i_3,i_3}, x_{i_1,i_1} = -x_{i_3,i_3}$, which implies that the diagonal is zero as well. So, for $n \geq 3$, we have $\mathrm{Z}(\mathfrak{g}) = 0$. The same holds for $n = 1$, trivially. When $n = 2$, we change the basis to make $B = y\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$, with $y \neq 0, x \neq 0$. Then, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$0 = A^T B + BA = y\begin{pmatrix} a & cx \\ b & dx \end{pmatrix} + y\begin{pmatrix} a & b \\ cx & dx \end{pmatrix},$$

which implies that (assume that char $\mathbb{F} \neq 2$): $a = d = 0$ and $b = -cx$. Given

$$C = \begin{pmatrix} 0 & -cx \\ c & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & -dx \\ d & 0 \end{pmatrix},$$

they clearly always commute. So, for $n = 2$, $\mathrm{Z}(\mathfrak{g}) = \mathfrak{g}$.

(4) The computation is very similar to 3. This time, our B is antisymmetric. Suppose $x \in \mathrm{Z}(so_n(\mathbb{F}))$; then, we have $x^T B + Bx = 0$ and $xz = zx$ for any $z \in \mathfrak{g}$. Let $y = (M_{ij} + M_{ji})$, and $z = B^{-1}y$, then, it is in $\mathfrak{g}$:

$$z^T B + Bz = y^T (B^{-1})^T (-B)^T + BB^{-1}y = y - y^T = 0$$

Multiplying the commutation relation by B, we have $0 = Bxz - Bzx = -x^T Bz - Bzx = xy + yx$. Since y is symmetric, this means that $yx = -(yx)^T$. Observe that yx is all zeros, except for a copy of jth row of x in the ith row place, and a copy of the ith row of x in the jth row place. Since yx is antisymmetric, all but four positions of x are forced to be 0. We get no constraint on $x_{i,j}$ and $x_{j,i}$, since these land on the diagonal in yx. However, we do get that $x_{i,i} = x_{j,j}$. Supposing that $n \geq 3$, this means we can zero out all the off-diagonal elements. Additionally, we have equality relations between any element of the diagonal, meaning that $x = aI$ (and we can pick any a). However, aI can only be in $\mathfrak{g}$ if $2aB = 0$. This is impossible, except in characteristic 2.

When $n = 2$, $B = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$, with $x \neq 0$. Then, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, to be in $\mathfrak{g}$. we must have

$$0 = A^T B + BA = x\begin{pmatrix} -c & a \\ -d & b \end{pmatrix} + x\begin{pmatrix} c & d \\ -a & -b \end{pmatrix}$$

so we see that $a = -d$. If $C \in \mathrm{Z}(\mathfrak{g})$, then we also know $a = d$ (from above). So, if C commutes with everything, it has the form $\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$. Then, the following must hold for all $A \in \mathfrak{g}$:

$$0 = AC - CA = \begin{pmatrix} bv & au \\ -av & cu \end{pmatrix} - \begin{pmatrix} cu & -au \\ av & bv \end{pmatrix} \Rightarrow C = 0.$$

The only antisymmetric matrix of dimension 1 is 0, so that case is nonsensical. Thus, the center of $sp_n(\mathbb{F})$ is always 0.