18.745 Introduction to Lie Algebras Fall 2004

Lecture 4: Nilpotent and Solvable Lie Algebras

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Definition Given a Lie algebra $\mathfrak{g}$, define the following descending sequences of ideals of $\mathfrak{g}$;

$$\text{(central series)} \quad \mathfrak{g} \supset [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}^1 \supset [[\mathfrak{g},\mathfrak{g}],\mathfrak{g}] = \mathfrak{g}^3 \supset [\mathfrak{g}^3,\mathfrak{g}] = \mathfrak{g}^4 \supset \cdots \supset [\mathfrak{g}^{k-1},\mathfrak{g}] = \mathfrak{g}^k \supset \ldots \tag{1}$$

$$\text{(derived series)} \quad \mathfrak{g} \supset [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}' \supset [\mathfrak{g}',\mathfrak{g}'] = \mathfrak{g}'' \supset \cdots \supset [\mathfrak{g}^{(k-1)},\mathfrak{g}^{(k-1)}] = g^{(k)} \supset \ldots \tag{2}$$

Exercise 4.1 Show that:
a) All members of the central and derived series are ideals of $\mathfrak{g}$
b) $\mathfrak{g}^{k+1} \supset \mathfrak{g}^{(k)}$

Solution a) First, let's show that $\mathfrak{g}^k$ is an ideal of $\mathfrak{g}$. Notice that:

$$[\mathfrak{g}^k,\mathfrak{g}] = \mathfrak{g}^{k+1} \subset \mathfrak{g}^k \tag{3}$$

It therefore follows that $\mathfrak{g}^k$ is an ideal for all k.
Now, let's show that $\mathfrak{g}^{(k)}$ is an ideal. To do this, we'll use induction on k. First notice that:

$$[\mathfrak{g}^{(1)},\mathfrak{g}] = [[\mathfrak{g},\mathfrak{g}],\mathfrak{g}] \subset [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}^{(1)} \tag{4}$$

this is true since we know $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$. It follows that $\mathfrak{g}^{(1)}$ is an ideal of $\mathfrak{g}$. Now all we have to prove is that $\mathfrak{g}^{(k)}$ is an ideal if $\mathfrak{g}^{(k-1)}$ is one. Therefore, consider an arbitrary element $x = [a,b] \in [\mathfrak{g}^{(k)},\mathfrak{g}]$ where $a \in \mathfrak{g}^{(k)}$ and $b \in \mathfrak{g}$. Then, by definition of $\mathfrak{g}^{(k)}$, $a = [c,d]$ where $c, d \in \mathfrak{g}^{(k-1)}$. Using the Jacobi identity, we find that:

$$[[c,d],b] = [[c,b],d] + [c,[d,b]] \tag{5}$$

but, since $\mathfrak{g}^{(k-1)}$ is an ideal, $[c,b],[d,b] \in \mathfrak{g}^{(k-1)}$. Therefore, it follows that:

$$[a,b] = [[c,d],b] = [[c,b],d] + [c,[d,b]] \in [\mathfrak{g}^{(k-1)},\mathfrak{g}^{(k-1)}] = \mathfrak{g}^{(k)} \tag{6}$$

Hence, since the element $x = [a,b]$ was arbitrary, $[\mathfrak{g}^{(k)},\mathfrak{g}] \subset \mathfrak{g}^{(k)}$ and therefore $\mathfrak{g}^{(k)}$ is an ideal for all k.

b)We need to show that $\mathfrak{g}^{k+1} \supset \mathfrak{g}^{(k)}$. Again, we'll use induction on k.

$$\mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}^2 \quad \Rightarrow \quad \mathfrak{g}^{(1)} \subset \mathfrak{g}^2 \tag{7}$$

Now consider an arbitrary element of $\mathfrak{g}^(k)$, $x = [a,b]$ where $a, b \in \mathfrak{g}^{(k-1)}$, we're assuming that since $a \in \mathfrak{g}^{(k-1)}$, $a \in \mathfrak{g}^k$ and also notice that $b \in \mathfrak{g}$. Therefore $[a,b] \in [\mathfrak{g}^k,\mathfrak{g}] = \mathfrak{g}^{k+1}$. Hence $\mathfrak{g}^{(k)} \subset \mathfrak{g}^{k+1}$ for all k. $\square$

Definition A Lie algebra is called *nilpotent* (resp. *solvable*) if $\mathfrak{g}^k = 0$ (resp. $\mathfrak{g}^{(k)} = 0$) for k sufficiently large.

Remark Notice that it follows from *Exercise 4.1b* that a nilpotent Lie algebra is also solvable.

Examples
1) Abelian Lie algebras are nilpotent.
2) The Heisenberg Lie Algebras $\mathcal{H}_k$ are nilpotent since $\mathcal{H}_k^{(1)} = \mathbb{C}$ and $[[\mathcal{H}_k, \mathcal{H}_k], \mathcal{H}_k] = 0$
3) $\mathfrak{b}_2 = \mathbb{F}a + \mathbb{F}b$, $[a, b] = 0$ is solvable. since ${\mathfrak{b}_2}^{(1)} = \mathbb{F}b$, $\mathfrak{b}_2^{(2)} = 0$ but it is not nilpotent since $\mathfrak{b}_2^n = \mathbb{F}b$ for $n \geq 2$.

Exercise 4.2 Let $\mathfrak{b}_k(\mathbb{F})$ be the subalgebra of $\mathfrak{gl}_k(\mathbb{F})$ consisting of all upper-triangular matrices and $\mathfrak{n}_k(\mathbb{F})$ be the subalgebra of $\mathfrak{gl}_k(\mathbb{F})$ consisting of all strictly upper triangular matrices. Show that:
a) $\mathfrak{b}_k(\mathbb{F})$ is a solvable Lie algebra (but not nilpotent)
b) $\mathfrak{n}_k(\mathbb{F})$ is a nilpotent Lie algebra (in particular, $\mathfrak{n}_3(\mathbb{F}) \approx \mathcal{H}_1$)

Solution The gist of this problem is to show that the Lie bracket of any two upper triangular matrices is even more upper triangular. It will actually prove easier to prove part (b) first.
b) First notice that if e_{ij} is a basis element of $\mathfrak{gl}_k(\mathbb{F})$, then the quantity $j - i$ tells us which diagonal the nonzero entry of e_{ij} lies on. Let's compute the Lie bracket $[e_{ij}, e_{mn}] = \delta_{jm} e_{in} - \delta_{ni} e_{mj}$. Let's say that $j - i = r$ and $n - m = s$. Then, when $j = m$, $n - i = n + r - j = n + r - m = s + r$, so $n - i = r + s$ and similarly if $n = i$, then $j - m = s + r$. The payoff from this little bit of arithmetic comes from the following observation. First, define:

$$\mathfrak{h}_{k,m} = \left\{ \sum_{i,j} x_{ij} e_{ij} | x_{ij} \in \mathbb{F}; j - i \geq m; i, j \leq k \right\} \tag{8}$$

Then, we've shown that $[\mathfrak{h}_{k,m}, \mathfrak{h}_{k,n}] \subset \mathfrak{h}_{k,m+n}$. Notice also that $\mathfrak{n}_k(\mathbb{F}) = \mathfrak{h}_{k,1}$ and $\mathfrak{b}_k(\mathbb{F}) = \mathfrak{h}_{k,0}$. Also notice from the definition that $\mathfrak{h}_{k,k+1} = 0$. Proving that $\mathfrak{n}_k(\mathbb{F})$ is nilpotent is now trivial:

$$\begin{aligned}
\mathfrak{n}_k(\mathbb{F})^2 &= \mathfrak{h}_{k,1}^2 \subset \mathfrak{h}_{k,2} \\
\mathfrak{n}_k(\mathbb{F})^3 &= [\mathfrak{h}_{k,1}^2, \mathfrak{h}_{k,1}] \subset [\mathfrak{h}_{k,2}, \mathfrak{h}_{k,1}] \subset \mathfrak{h}_{k,3} \\
&\vdots \\
\mathfrak{n}_k(\mathbb{F})^k &= \mathfrak{h}_{k,1}^k \subset \mathfrak{h}_{k,k+1} = 0
\end{aligned}$$

Therefore $\mathfrak{n}_k(\mathbb{F})$ is nilpotent. Now, on to proving $\mathfrak{b}_k(\mathbb{F})$ is solvable.
a) To prove this, we will use a proposition that will appear later in these notes. The proposition states that if a Lie algebra contains a solvable ideal and if the Lie algebra modulo this ideal is solvable, then the ideal itself is solvable. Notice that $\mathfrak{n}_k(\mathbb{F}) \subset \mathfrak{b}_k(\mathbb{F})$ and that $\mathfrak{n}_k(\mathbb{F})$ is solvable (since it's nilpotent). Also notice that $\mathfrak{b}_k(\mathbb{F})/\mathfrak{n}_k(\mathbb{F}) =$ the set of diagonal matrices, which is abelian, hence solvable. Therefore, by the proposition, $\mathfrak{b}_k(\mathbb{F})$ is solvable.
Notice that in particular,

$$\mathfrak{n}_3(\mathbb{F}) = span \left\{ q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \tag{9}$$

These matrices exactly satisfy the conditions for the Heisenberg algebra $\mathcal{H}_1$. Hence $\mathfrak{n}_3(\mathbb{F}) \approx \mathcal{H}_1$.

Also, we have proved that $\mathfrak{b}_k(\mathbb{F})$ is solvable, but we have yet to show that is not nilpotent. To demonstrate this, consider the commutator: $[e_{ii}, e_{ij}] = e_{ij}$ for $i < j$. Since e_{ii} is in $\mathfrak{b}_k(\mathbb{F})$, it follows that $e_{ij} \in \mathfrak{b}_k(\mathbb{F})^2$, and therefore, $e_{ij} \in [\mathfrak{b}_k(\mathbb{F}), \mathfrak{b}_k(\mathbb{F})^2] = \mathfrak{b}_k(\mathbb{F})^3$. Hence $e_{ij} \in \mathfrak{b}_k(\mathbb{F})^k$ for all k follows by induction. Thus $\mathfrak{b}_k(\mathbb{F})$ is not nilpotent. □

Remark Notice that any subalgebra and factor (quotient) algebra of a nilpotent (resp. solvable) Lie algebra is a nilpotent (resp. solvable) Lie algebra.

Proposition 1) The center of a nonzero nilpotent Lie algebra $\mathfrak{g}$ is nonzero.
2) If $\mathfrak{g}/\text{center}(\mathfrak{g})$ is a nilpotent Lie algebra, then $\mathfrak{g}$ is a nilpotent Lie algebra.

Proof a) Let $k \geq 2$ be the minimal integer such that $\mathfrak{g}^k = 0$. Then $\mathfrak{g}^{k-1} \neq 0$ and $[\mathfrak{g}^{k-1}, \mathfrak{g}] = \mathfrak{g}^k = 0$. Hence $\mathfrak{g}^{k-1} \subset \mathit{center}(\mathfrak{g})$
b) If $\mathfrak{g}/\text{center}(\mathfrak{g})$ is a nilpotent Lie algebra then any commutator in $\mathfrak{g}/\text{center}(\mathfrak{g})$ of length k ($k \gg 0$) is zero:

$$0 = [[\dots[a_1 + e, a_2 + e], a_3 + e], \dots, a_k + e] \tag{10}$$
$$= [\dots[a_1, a_2], a_3], \dots, a_k] + e \tag{11}$$

Hence in $\mathfrak{g}$, we have: $[\dots[a_1, a_2], \dots, a_k] \in e$, where e denotes the center of $\mathfrak{g}$. Hence any commutator of length $k+1$ in $\mathfrak{g}$ is zero, in other words, $\mathfrak{g}^{k+1} = 0$.

Theorem (*The Engel Characterization of Nilpotent Lie Algebras*)
A finite dimensional Lie algebra $\mathfrak{g}$ is nilpotent if and only if ad a is a nilpotent operator for any $a \in \mathfrak{g}$.

Proof If $\mathfrak{g}$ is nilpotent, then for $k \gg 0$, any commutator of length k is zero, in particular, $(\text{ad } a)^k b = 0$, being a commutator of length $k+1$.
Conversely, suppose $(\text{ad } a)^k = 0$ for all $a \in \mathfrak{g}$, $k \gg 0$. Conside the adjoint representation defined by:

$$\mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}} \tag{12}$$
$$a \mapsto \text{ad } a \tag{13}$$

Then ad $\mathfrak{g} \subset \mathfrak{gl}_{\mathfrak{g}}$ is isomorphic to $\mathfrak{g}/\mathit{center}(\mathfrak{g})$ since $\mathit{center}(\mathfrak{g}) = \ker \text{ad}$. So it suffices to prove that ad $\mathfrak{g}$ is a nilpotent Lie algebra. Now, we can apply Engel's Theorem: all operators from ad $\mathfrak{g} \subset \mathfrak{gl}_{\mathfrak{g}}$ are nilpotent, hence ad $\mathfrak{g} \subset \mathfrak{n}_{\mathfrak{g}}$ (the Lie algebra of strictly upper-triangular matrices, for some choice of a basis of $\mathfrak{g}$), and so ad $\mathfrak{g}$ is a subalgebra of a nilpotent Lie algebra, hence is a nilpotent Lie algebra itself. In particular, we see that $(\text{ad } a)^{dim \mathfrak{g}} = 0$.

Discussion What follows is a discussion of the classification of finite dimensional nilpotent Lie algebras.

Consider only the 2-step nilpotent Lie algebras: $\mathfrak{g} \supset \mathfrak{e}$ where $\mathfrak{e} = \mathit{center}(\mathfrak{g})$ and $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{e}$ is abelian (if and only if $\mathfrak{g}^3 = 0$).
We have the following invariant of $\mathfrak{g}$: the map φ,

$$\varphi : \bar{\mathfrak{g}} \otimes \bar{\mathfrak{g}} \to e \tag{14}$$

$$\varphi(\bar{a} \otimes \bar{b}) = [a, b] \tag{15}$$

Where a and b are the pre-images in $\mathfrak{g}$ of $\bar{a}$ and $\bar{b}$. The only condition on φ is skew-symmetry:

$$\varphi(\bar{b} \otimes \bar{a}) = -\varphi(\bar{a} \otimes \bar{b}) \tag{16}$$

So we have a bijective correspondence between the set of 2-step nilpotent Lie algebras and the set of skew-symmetric maps ($\varphi : \bar{\mathfrak{g}} \otimes \bar{\mathfrak{g}} \to \mathit{center}(\mathfrak{g})$).

Exercise 4.3 Show that if $\mathfrak{g}$ is 2-step nilpotent and dim center$(\mathfrak{g}) = 1$, then $\mathfrak{g} \approx \mathcal{H}_k$.

Solution By definition, $\mathfrak{g}$ is nilpotent if $\mathfrak{g}/\mathit{center}(\mathfrak{g})$ is abelian. Therefore, $[\mathfrak{g}, \mathfrak{g}] \subset \mathit{center}(\mathfrak{g})$. If the dimension of the center of $\mathfrak{g}$ is one, the we can conclude that the derived algebra of $\mathfrak{g}$also has dimension one. Therefore, by a Exercise 2.3, we can conclude that: $\mathfrak{g} = \mathfrak{b}_2 \oplus ab_{n-2}$ or $\mathfrak{g} = \mathcal{H}_k \oplus ab_{n-2k-1}$ where ab_n denotes the n-dimensional Lie abelian algebra. However, $\mathfrak{b}_2 \oplus ab_{n-2}$ is not nilpotent. Therefore $\mathfrak{g} = \mathcal{H}_k \oplus ab_{n-2k-1}$. However, we know that dim($\mathit{center}(\mathfrak{g})$)= 1, hence $\mathfrak{g} = \mathcal{H}_k$. $\square$

Remark As discussed in lecture, the next case, dim center$(\mathfrak{g})$ is a "wild" problem and the general classification of nilpotent Lie algebras is believed to be impossible, if not very difficult.

Proposition Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$. Then $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}/\mathfrak{h}$ is solvable and $\mathfrak{h}$ is solvable.

Remark This is not true if "solvable" is replaced by "nilpotent", as the case of $\mathfrak{b}_2$, where $[a, b] = b$, demonstrates.

Proof $\mathfrak{g}$ is solvable, therefore $\mathfrak{h}$ is solvable and $\mathfrak{g}/\mathfrak{h}$ is solvable.
Conversely, if $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{h}$ is solvable, then $\mathfrak{g}^{(k)} \subset \mathfrak{h}$. However, if in addition, $\mathfrak{h}$ is solvable (i.e. $\mathfrak{h}^{(l)} = 0$ for some $l \geq 1$), then $\mathfrak{g}^{(k+l)} \subset \mathfrak{h}^{(l)} = 0$, so $\mathfrak{g}^{(k+l)} = 0$. Hence $\mathfrak{g}$ is solvable.