18.745: Lecture 5

Professor: Victor Kač
Scribe: Zhongtao Wu

Definition 1. If $\mathfrak{h}$ is a Lie algebra and $\pi : \mathfrak{h} \to gl_V$ is its representation, and $\lambda \in \mathfrak{h}^*$, then the subspace $V_\lambda^{\mathfrak{h}} = \{v \in V | \pi(a)v = \lambda(a)v \text{ for all } a \in \mathfrak{h}\}$ is called a weight space of $\mathfrak{h}$ attached to λ.

Lemma 2 (Lie's lemma). *Let $\mathbb{F}$ be a field of characteristic 0; let π be a representation of $\mathfrak{g}$ in a finite-dimensional vector space V, and let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$. Then for any $\lambda \in \mathfrak{h}^*$, $\pi(\mathfrak{g})V_\lambda^{\mathfrak{h}} \subset V_\lambda^{\mathfrak{h}}$. (i.e weight spaces of $\mathfrak{h}$ are $\mathfrak{g}$-invariant.)*

Proof. We may assume $V_\lambda^{\mathfrak{h}} \neq 0$. We want to prove that if $\pi(g)v \in V_\lambda^{\mathfrak{h}}$, then $\pi(g)v \in V_\lambda^{\mathfrak{h}}$, namely:

$$\pi(h)(\pi(g)v) = \lambda(h)\pi(g)v, \ \forall h \in \mathfrak{h}, g \in \mathfrak{g}, v \in V_\lambda^{\mathfrak{h}}$$

Using $AB = BA + [A, B]$, we can rewrite the equation as:

$$\pi(g)\pi(h)v + \pi([h, g])v = \lambda(h)\pi(g)v.$$

Since $\mathfrak{h}$ is an ideal, $[h, g] \in \mathfrak{h}$. So we can use the definition of $V_\lambda^{\mathfrak{h}}$ to further simplify the equation to:

$$\lambda(h)\pi(g)v + \lambda([h, g])(v) = \lambda(h)\pi(g)v$$

or equivalently $\lambda([h, g])(v) = 0$. So we need to show that if $V_\lambda^{\mathfrak{h}} \neq 0$, then $\lambda([h, g]) = 0$ for all $h \in \mathfrak{h}, g \in \mathfrak{g}$.

Pick a non-zero vector $v \in V_\lambda^{\mathfrak{h}}$ and take any $g \in \mathfrak{g}$. Denote by W_m the span of vectors $v, \pi(g)v, \ldots, \pi(g)^m v$. We have an increasing sequence of subspaces $W_0 = \mathbb{F}v \subset W_1 = \mathbb{F}v + \mathbb{F}\pi(g)v \subset \cdots$ in V. Take the smallest N, such that the vectors $v, \pi(g)v, \ldots, \pi(g)^{N-1}v$ are linearly independent, but $v, \ldots, \pi(g)^N v$ are linearly dependent, i.e, $\pi(g)^N v =$ linear combination of $v, \ldots, \pi(g)^{N-1}v$; and the same holds for $\pi(g)^{N+1}v, \ldots$. Hence,

$$\pi(g)W_{N-1} \subset W_{N-1} = W_N = W_{N+1} = \cdots$$

We shall prove that for any $h \in \mathfrak{h}$, $\pi(h)W_m \subset W_m$ for all m less than N, and moreover, that in the basis $v, \pi(g)v, \ldots, \pi(g)^m v$, the operator $\pi(h)$ has the matrix form $\begin{pmatrix} \lambda(h) & * & \\ & \ddots & \\ 0 & & \lambda(h) \end{pmatrix}$

We do it by induction on n: For $n = 0$, we have matrix $(\lambda(h))$. Suppose this is true for $< n$, then:

$$\pi(h)\pi(g)^n v = \pi(h)\pi(g)\pi(g)^{n-1} v = \pi([h, g])\pi(g)^{n-1} v + \pi(g)\pi(h)\pi(g)^{n-1} v.$$

Since $[h, g] \in \mathfrak{h}$, the first term $\pi([h, g])\pi(g)^{n-1} \in W_{n-1}$. For our second term, we use the induction hypothesis on $\pi(h)\pi(g)^{n-1} v$ to get:

$$\begin{aligned}\pi(g)\pi(h)\pi(g)^{n-1} v &= \pi(g)(\lambda(h)\pi(g)^{n-1} v + c_{n-2}\pi(g)^{n-2} v + \cdots + c_0 v) \\ &= \lambda(h)\pi(g)^n v + c_{n-2}\pi(g)^{n-2} v + \cdots + c_0 \pi(g) v\end{aligned}$$

So we proved the inductive step. In particular, both $\pi(h)$ and $\pi(g)$ are operators on W_{N-1} and $\pi(h)$ is upper triangular on W_{N-1} with $\lambda(h)$ on the diagonal. But $\pi([h, g]) = [\pi(h), \pi(g)]$ on W_{n-1}; if we take $\operatorname{tr}_{W_{N-1}}$ of both sides, we get $N\lambda([h, g]) = 0$. So $\lambda([h, g]) = 0$ since $\operatorname{char} \mathbb{F} = 0$.

□

From the above lemma, we can easily prove the following main theorem in this section.

Theorem 3 (Lie's theorem). *Let $\mathbb{F}$ be an algebraically closed field of characteristic 0. Let $\mathfrak{g}$ be a solvable Lie algebra and let $\pi : \mathfrak{g} \to gl_V$ be a representation of $\mathfrak{g}$ in a finite dimensional vector space V over $\mathbb{F}$. Then there exists a common eigenvector $v \in V$ for all operators $\pi(a)$, $a \in \mathfrak{g}$. i.e, $\pi(a)v = \lambda(a)v$, where $\lambda(a) \in \mathbb{F}$, $v \neq 0$.*

Proof. Note that $\pi(g) \subset gl_V$ is a finite dimensional subalgebra since $\dim gl_V = (\dim V)^2$, so we may assume that $\dim \mathfrak{g} < \infty$ and prove Lie's theorem by induction on $n = \dim \mathfrak{g}$.

If $n = 1$, i.e, $\mathfrak{g} = \mathbb{F}a$, then we take for v any eigenvector of $\pi(a)$. (We use here that $\mathbb{F}$ is algebraically closed.)

For any $n > 1$, since $\mathfrak{g}$ is solvable, $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$; so we can take a subspace $\mathfrak{h}$ of $\mathfrak{g}$ of codimension 1 containing $[\mathfrak{g}, \mathfrak{g}]$. Then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ since $[\mathfrak{h}, \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$. $\dim \mathfrak{h} = n - 1$, so we can apply the inductive assumption and find $v \in V$ such that $\pi(h)v = \lambda(h)v$ for all $h \in \mathfrak{h}$. Obviously $\lambda \in \mathfrak{h}^*$, so $V_\lambda^{\mathfrak{h}} \neq 0$. Apply Lie's lemma, we get $\pi(a)V_\lambda^{\mathfrak{h}} \subset V_\lambda^{\mathfrak{h}}$ for all $a \in \mathfrak{g}$. Write $\mathfrak{g} = \mathfrak{h} + \mathbb{F}a$, we have $\pi(a)V_\lambda^{\mathfrak{h}} \subset V_\lambda^{\mathfrak{h}}$. But $V_\lambda^{\mathfrak{h}}$ is a finite dimensional vector space over an algebraically closed field, hence the operator $\pi(a)$ has an eigenvector $v \in V_\lambda^{\mathfrak{h}}$. This is the desired v. □

Exercise 5.1: Lie's lemma and hence Lie's theorem hold over an algebraically closed field of $\operatorname{char} \mathbb{F} = p > \dim V$.

Proof. The only place we used $\operatorname{char} \mathbb{F} = 0$ in our proof of Lie's lemma is concluding $\lambda([\mathfrak{h}, \mathfrak{g}]) = 0$ from the identity $N\lambda([\mathfrak{h}, \mathfrak{g}]) = 0$, where N is the dimension of the subspace W_{N-1}. But if $\operatorname{char} \mathbb{F} = p > \dim V > \dim W_{N-1} = N$, such an argument is still valid. □

Exercise 5.2: Take $\mathfrak{g} = \mathfrak{H}_1$ acting on $\mathbb{F}[x]$ by $P = \frac{d}{dx}$, $Q = x$, $c = 1$. Suppose $\operatorname{char}\mathbb{F} = p > 0$, then $U = \oplus_{j \geq p}\mathbb{F}x^j$ is a subrepresentation. Hence $V = \mathbb{F}[x]/U$ is a p-dimensional representation of $\mathfrak{H}_1$. It can be shown that there is no common eigenvector for P and Q, so Lie's theorem fails.

Proof. Suppose $f \in U$, then $Q(f) \in U$. If $\deg f > p$, then $\deg P(f) \geq p$ and hence $P(f) \in U$. Otherwise, $\deg f = p$, i.e, $f = cx^p$ for some constant c. Then $P(f) = cpx^{p-1} = 0$ since $\operatorname{char}\mathbb{F} = p$. Hence, U is a subrepresentation, and the quotient $V = \mathbb{F}[x]/U$ is a p-dimensional representation of $\mathfrak{g}$.

Next, notice that the only nonzero eigenvector for Q is a scalar of x^{p-1}. But x^{p-1} is not an eigenvector for P, since $P(x^{p-1}) = (p-1)x^{p-2}$. So Lie's theorem fails. □

Exercise 5.3: If $\mathfrak{g}$ is abelian, then $\lambda([h, g]) = 0$ for all $\lambda \in \mathfrak{h}^*$, $g, h \in \mathfrak{g}$. So Lie's lemma and hence Lie's theorem hold over any algebraically closed field, even if $\operatorname{char}\mathbb{F} \neq 0$.

Proof. If $\mathfrak{g}$ is abelian, then the commutator $[\pi(g), \pi(h)]$ is identically 0. So we have

$$\pi(h)(\pi(g)v) = \pi(g)\pi(h)v = \lambda(h)(\pi(g)v), \ \forall h \in \mathfrak{h}, g \in \mathfrak{g}.$$

This means that $\pi(g)V_\lambda^{\mathfrak{h}} \subset V_\lambda^{\mathfrak{h}}$, hence Lie's lemma holds. □

Lie's theorem implies the following corollary:

Corollary 4. *(a) For any representation π of a solvable Lie algebra $\mathfrak{g}$ in a finite dimensional vector space V, there exists a basis of V for which the matrices of all operators $\pi(g)$, $g \in \mathfrak{g}$ are upper triangular.*

(b) A subalgebra $\mathfrak{g} \subset gl_V$, $\dim V < \infty$, is solvable if and only if in some basis the matrices of all operators from $\mathfrak{g}$ are upper triangular.

(c) If $\mathfrak{g}$ is a finite dimensional solvable Lie algebra, then $[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent Lie algebra.

Proof. (a) By Lie's theorem, there exists a common eigenvector v for $\pi(\mathfrak{g})$. Let $v_1 = v$. The subspace $\mathbb{F}v_1$ is $\pi(\mathfrak{g})$ invariant, hence we may consider the representation $\pi_{V/\mathbb{F}v_1}$ of $\mathfrak{g}$ in $V/\mathbb{F}v_1$. Apply Lie's theorem, we can find a common eigenvector $v_2' \in V/\mathbb{F}v_1$ for $\pi_{V/\mathbb{F}v_1}(\mathfrak{g})$. This means that if $v_2 \in V$ is a preimage of v_2', then $\pi(\mathfrak{g})v_2 \in \mathbb{F}v_1 + \mathbb{F}v_2$. Next, consider $V/(\mathbb{F}v_1 + \mathbb{F}v_2)$ and construct $v_3 \in V$ such that $\pi(\mathfrak{g})v_3 \in \mathbb{F}v_1 + \mathbb{F}v_2 + \mathbb{F}v_3$, etc.

So for any $a \in \mathfrak{g}$,

$$\begin{aligned}
&\pi(a)v_1 \in \mathbb{F}v_1\\
&\pi(a)v_2 \in \mathbb{F}v_1 + \mathbb{F}v_2\\
&\pi(a)v_3 \in \mathbb{F}v_1 + \mathbb{F}v_2 + \mathbb{F}v_3\\
&\quad \cdots
\end{aligned}$$

But this exactly means that $\pi(a)$ is upper triangular in the basis $v_1, \ldots, v_n$.

(b) If the matrices of all operators from $\mathfrak{g}$ are upper triangular in some basis, then it is a subalgebra of the solvable Lie algebra $b_k(\mathbb{F})$, which consists of all upper triangular matrices. Hence, $\mathfrak{g}$ is also solvable.

Conversely, if $\mathfrak{g} \subset gl_V$ is solvable, then we can apply part (a) to find a basis in which all matrices are upper triangular.

(c) Consider $\operatorname{ad} : \mathfrak{g} \to gl_{\mathfrak{g}}$, the adjoint representation of $\mathfrak{g}$. $\operatorname{Ker}\operatorname{ad} = \operatorname{center}(\mathfrak{g})$ is abelian, hence is a solvable ideal. Also, $\operatorname{ad}(\mathfrak{g}) = \mathfrak{g}/\operatorname{center}(\mathfrak{g}) \subset gl_{\mathfrak{g}}$ and $\operatorname{ad}([\mathfrak{g}, \mathfrak{g}]) = [\mathfrak{g}, \mathfrak{g}]/[\mathfrak{g}, \mathfrak{g}] \cap \operatorname{center}(\mathfrak{g}) \subset gl_{\mathfrak{g}}$. In order to prove $[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent Lie algebra, it suffices to prove $\operatorname{ad}[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. But $\operatorname{ad}\mathfrak{g} \subset gl_{\mathfrak{g}}$ is a solvable subalgebra, hence is an upper triangular subalgebra of $gl_{\mathfrak{g}}$ in some basis of $\mathfrak{g}$ by part (b). But the commutator of two upper triangular matrices is strictly upper triangular, hence $\operatorname{ad}[\mathfrak{g}, \mathfrak{g}] \subset$ strictly upper triangular. So $\operatorname{ad}[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent Lie algebra; hence $[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent Lie algebra.

$\square$