18.745 Introduction to Lie algebra
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Lecture 6 , noted by Liu Ruochuan

Throughout this lecture $\mathbb{F}$ will be assumed to be an algebraically closed field.

Jordan Decomposition

Now suppose V is a finite dimensional vector space over $\mathbb{F}$ and A is a linear operator on V. Let $\{\lambda_1, \cdots, \lambda_s\}$ denote the set of distinct eigenvalues of A. By the theorem of Jordan normal form, in some basic of V, the matrix of A is a direct sum of the Jordan blocks J_{λ_i} assigned to some eigenvalue λ_i. If we denote by V_{λ_i} the span of the vectors in the basis which correspond to all Jordan blocks assigned to λ_i, we obtain the following generalized eigenspace decomposition of V.

$$V = \bigoplus_{i=1}^{s} V_{\lambda_i},$$

where the generalized eigenspace V_{λ_i} can also be defined as

$$V_{\lambda_i} = \{v \in V | (A - \lambda_i I)^N v = 0 \text{ for some } N \ \}$$

We take A_s as the diagonal part of the Jordan normal form of A, $A_n = A - A_s$. Then A_s is semisimple, i.e diagonalizable operator and A_n is a nilpotent operator. Moreover, $A_s A_n = A_n A_s$. This decomposition is called a Jordan decomposition of A.

Ex6.1 Show that there exist polynomials $P(x)$ and $Q(x)$ such that $A_s = P(A)$ and $A_n = Q(A)$
Solution. By Chinese remainder theorem, there exists a polynomial $P(x)$ such that $P(x) \equiv \lambda_i (mod (x - \lambda_i)^n)$ for every λ_i, where $n = \dim V$. Then for $v \in V_{\lambda_i}$, we have $P(A)v = \lambda_i v$. That means $P(A) = A_s$ and $(1 - P)(A) = A_n$.

Ex6.2 If linear operators A and B commute, then any eigenspace and generalized eigenspace of A is B-invariant. Conclude that two commuting semi-simple operators can be diagonalized in the same basic.
Solution. Let V_{λ_i} be the generalized eigenspace of A with eigenvalue λ_i. For every $v \in V_{\lambda_i}$, we have $(A - \lambda_i)^N v = 0$ for some N. Since A commutes with B, we get $(A - \lambda_i)^N Bv = B(A - \lambda_i)^N v = 0$. Thus Bv is in V_{λ_i} by definition. That just means the generalized eigenspace of A is B-invariant. The method for eigenspace case is the same. Now suppose A and B are both semi-simple. Let $V = \bigoplus_{i=1}^{s} V_{\lambda_i}$ be the eigenspace decomposition of A. Since B is semi-simple and V_{λ_i} is an invariant subspace w.r.t B, B is semi-simple on V_{λ_i}. Then B can be diagonalized in V_{λ_i} under some basis. Now under the basis, which is the union of these basis, A and B are all diagonal.

The Jordan decomposition of a linear operator A is unique in sense of

Theorem 1. *Let $A = A_s' + A_n'$ where A_s' and A_n' are also linear operators which satisfy*
(1)A_s' is diagonalizable
(2)A_n' is nilpotent
(3)$A_s'A_n' = A_n'A_s'$
Then $A_s' = A_s$, $A_n' = A_n$.

Proof. We first have $A_s' - A_s = A_n - A_n'$. Note that A_s' and A_n' commute with A. So by Ex 6.1, A_s' and A_n' commute with both A_s and A_n. Then by Ex 6.2, $A_s' - A_s$ is semisimple. But $A_n - A_n'$ is nilpotent by the binomial formula. Since the only nilpotent semisimple operator is 0, we conclude that $A_s' = A_s$ and $A_n' = A_n$ from the equation given above. □

Generalized Weight Space Decomposition

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{F}$ and π a representation of $\mathfrak{g}$ in a finite dimensional vector space V over $\mathbb{F}$. Consider the generalized eigenspace decomposition of $\mathfrak{g}$ w.r.t ad$a$ and of V w.r.t $\pi(a)$.

$$\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha^a \text{ where } \mathfrak{g}_\alpha^a = \{g \in \mathfrak{g} | (\mathrm{ad}a - \alpha)^N g = 0 \text{ for some } N\}$$

$$V = \bigoplus_\lambda V_\lambda^a \text{ where } V_\lambda^a = \{v \in V | (\pi(a) - \lambda)^N v = 0 \text{ for some } N\}$$

These two decompositions are related by

Proposition 1. $\pi(\mathfrak{g}_\alpha^a)V_\lambda^a \subseteq V_{\lambda+\alpha}^a$

We need the following lemma to finish the proof.

Lemma 1. *Let A be a unital associate algebra over $\mathbb{F}$. Let a, $b \in A$ and α, $\lambda \in \mathbb{F}$. Then we have the following identity*

$$(a - \alpha - \lambda)^N b = \sum_{j=0}^N \binom{N}{j} (ada - \alpha)^j b (a - \lambda)^{N-j}$$

Proof. Let L_x and R_x denote the operators of left and right multiplication in A by x. Then associativity means that $L_x R_y = R_y L_x$. We have $L_{a-\alpha-\lambda} = (\mathrm{ad}a - \alpha) + R_{a-\lambda}$. Note that ad$a - \alpha$ commute with $R_{a-\lambda}$, hence $L_{a-\alpha-\lambda}^N = \sum_{j=0}^N \binom{N}{j} (\mathrm{ad}a - \alpha)^j R_{\alpha-\lambda}^{N-j}$ by the binomial formula. Now apply both sides to b to obtain the identity. □

Proof of the proposition. Suppose $g \in \mathfrak{g}_\alpha^a$, $v \in V_\lambda^a$. Apply the lemma to A =End(V), a is $\pi(a)$, $b = \pi(g)$, hence

$$(\pi(a) - \alpha - \lambda)^N \pi(g) v = \sum_{j=0}^N \binom{N}{j} (\mathrm{ad}\pi(a) - \alpha)^j \pi(g)(\pi(a) - \lambda)^{N-j} v$$

Take $N >$dim$\mathfrak{g}_\alpha^a$+dimV_λ^a, then each summand of the right hand side of the above equation is zero. That means the desired result.

Suppose $\mathfrak{h}$ is a finite dimensional Lie algebra over $\mathbb{F}$. Let π be a representation of $\mathfrak{h}$ in a finite dimensional vector space V over $\mathbb{F}$ and let $\lambda \in \mathfrak{h}^*$. Then the generalized weight space attached to λ is defined as

$$V_\lambda = \{v \in V | (\pi(h) - \lambda(h))^N v = 0 \text{ for every } h \in \mathfrak{h} \text{ and some } N\}.$$

Theorem 2. *Notation as above and further assume char.*$\mathbb{F} = 0$ *and* $\mathfrak{h}$ *is nilpotent. Then*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

Ex 6.3 a)Deduce the theorem from Ex 6.2 in the case when $\mathfrak{h}$ is abelian.
b)Consider the adjoint representation of the unique 2-dimensional non-abelian Lie algebra to show that theorem 2 fails if $\mathfrak{h}$ is not nilpotent.
Solution. a) For any $a \in \mathfrak{h}$, $\pi(\mathfrak{h})V_\lambda^a \subset V_\lambda^a$ by Ex 6.2 . So if there is at least one $\pi(a)$ has distinct eigenvalues, then we can apply induction on dim$V$. Otherwise we just need to prove that the only eigenvalue of every $\pi(a)$ is linear functional on $\mathfrak{h}$. Let a and b are elements of $\mathfrak{h}$. Suppose eigenvalues of $\pi(a)$ and $\pi(b)$ are λ and μ respectively. Then since $\mathfrak{h}$ is abelian, we have

$$(\pi(a+b) - \lambda - \mu)^N = \sum_{j=0}^{N} \binom{N}{j} (\pi(a) - \lambda)^j (\pi(b) - \mu)^{N-j}$$

Now choose $N > 2n$ where $n =$dim$V$. Then apply the above equation to every $v \in V$ to conclude that the eigenvalue of $\pi(a+b)$ is $\lambda + \mu$.
b)Denote the unique 2-dimensional non-abelian Lie algebra by $\mathfrak{b} = \mathbb{F}a + \mathbb{F}b$ which satisfies $[a, b] = b$. It is easy to see that $\mathbb{F}a$ is a generalized eigenspace of ad$a$, but it is not a generalized eigenspace of b. Thus the theorem may fails if the given Lie algebra is not nilpotent.

Proof. Take any $a \in \mathfrak{h}$, then $\mathfrak{h} = \mathfrak{h}_0^a$. Hence by the proposition which we just proved $\pi(\mathfrak{h})V_\lambda^a \subset V_\lambda^a$ for all $a \in \mathfrak{h}$ and eigenvalue λ of a. So if there is at least one $\pi(a)$ has distinct eigenvalues, then we can apply induction on dim$V$. Otherwise we just need to prove that the only eigenvalue of every $\pi(a)$ is linear functional on $\mathfrak{h}$. Apply Lie's theorem that all $\pi(a)$ are upper triangular in some basis, then we deduce the desired result since the eigenvalue are just the numbers on diagonal. □

Remark It is easy to see that we can still get the decomposition even if char.$\mathbb{F} \neq 0$. But λ may not be linear functional on $\mathfrak{h}$ in this case.

Ex 6.4 Consider the 2-dimensional representation of H_1 for char.$\mathbb{F} = 2$

$$V = \mathbb{F}[x]/x^2\mathbb{F}[x],\ p = \frac{\partial}{\partial x},\ q = x,\ c = 1.$$

Then $V = V_\lambda$, but λ is not a linear function.
Solution. Note that p and q are all nilpotent. But $p + q$ is not nilpotent since $(p+q)^2(x) = x$. So in this case λ is not a linear function.

Ex 6.5* If $\mathfrak{h}^p = 0$, then the theorem still holds for char.$\mathbb{F} = p$.