18.745 Introduction to Lie Algebras Fall 2004

Lecture 7 — September 30, 2004

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In the course of this lecture, $\mathbb{F}$ denotes an algebraically closed field of characteristic 0, $\mathfrak{g}$ denotes a finite dimensional Lie algebra over $\mathbb{F}$, $\mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}$, and π is a representation of $\mathfrak{h}$ in a finite dimensional vector space V over $\mathbb{F}$.

Last time we proved the validity of the *generalized weight space decomposition:*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda,$$

where V_λ is the generalized eigenspace $V_\lambda = \{v \in V | (\pi(h) - \lambda(h))^N v = 0 \text{ for } N >> 0\}$. In particular, taking the adjoint representation on $\mathfrak{g}$, we get the *generalized root space decomposition:*

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{g}^*} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the generalized rootspace $\mathfrak{g}_\alpha = \{g \in \mathfrak{g} | (ad(h) - \lambda(h))^N g = 0 \text{ for } N >> 0\}$. The reasons for calling such a decomposition a root space decomposition are historic. A relation between these two decompositions is given by $\pi(\mathfrak{g}_\alpha)V_\lambda \subset V_{\alpha+\lambda}$, which follows from a proposition we proved in lecture 6, namely that $\pi(\mathfrak{g}_\alpha^a)V_\lambda^a \subset V_{\alpha+\lambda}^a$. Futhermore, considering π to be the adjoint representation we obtain that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. These two relations play a very important role in the structure and representation theory of Lie algebras.

A digression to topological spaces

Definition 1 *A* topological space *is a set X together with a collection of its* closed subsets*, subject to the following axioms:*

(i) X and $\emptyset$ are closed

(ii) the union of any finite collection of closed subsets is closed

(iii) the intersection of any collection of closed subsets is closed

(iv) (weak separation axiom) given $x, y \in X$, $x \neq y$, there exists a closed subset F such that $x \in F$ and $y \notin F$.

Definition 2 *A set $U \subset X$ is called* open *if there exists a closed set V such that $U = X \setminus V$.*

Note that the weak separation axiom means that for any $x \neq y$ from X there exists an open set U such that $x \in U$ and $y \notin U$.

Definition 3 *The* Zariski topology *is a topology defined on $X = \mathbb{F}^n$ such that a closed subest is the set of common zeros of a set of polynomials in n indeterminates $\{P_\alpha(x)\}_{\alpha \in I}$, where I is some index set that could be infinite.*

Exercise 7.1. Prove that the Zariski topology is a topology.

Given a set S of polynomials we denote by $\mathbb{V}(S) \subset \mathbb{F}^n$ the set of common zeros of the polynomials in S. The notation $\mathbb{V}$ stands for *variety.* Expressed with this new notation all the closed subests of the Zariski topology on $\mathbb{F}^n$ are of the form $\mathbb{V}(S)$ for some set S of polynomials. A special case of a variety is a *hypersurface* $\mathbb{V}(P)$ where P is a given nonconstant polynomial. Note that by definition, any closed subset which is not the whole $\mathbb{F}^n$ lies in some hypersurface.

Solution to Ex.7.1. We need to check the four axioms for a topological space:

(a) X and $\emptyset$ are closed since $\mathbb{V}(0) = X$, and $\mathbb{V}(1) = \emptyset$

(b) The union of any finite collection of closed sets is closed:

Let $\mathbb{V}(S_1)$ and $\mathbb{V}(S_2)$ be two closed sets and let $S = \{f_1 f_2 | f_1 \in S_1 \text{and } f_2 \in S_2\}$. Then, if $x \in \mathbb{V}(S_1) \cup \mathbb{V}(S_2)$, then for any $f_1 \in S_1$, $f_2 \in S_2$, $f_1 f_2(x) = 0$, as either $f_1(x)$ or $f_2(x)$ is zero. Thus $\mathbb{V}(S_1) \cup \mathbb{V}(S_2) \subset \mathbb{V}(S)$.

Conversely, if $x \in \mathbb{V}(S)$ and $x \notin \mathbb{V}(S_1)$, then there is an $f_1 \in \mathbb{V}(S_1)$ such that $f_1(x) \neq 0$, and so $f_2(x) = 0$ for all $f_2 \in S_2$, thus $x \in \mathbb{V}(S_2)$, therefore $\mathbb{V}(S) \subset \mathbb{V}(S_1) \cup \mathbb{V}(S_2)$.

We have obtained that $\mathbb{V}(S) \subset \mathbb{V}(S_1) \cup \mathbb{V}(S_2)$, and this solves the problem as we can now perform induction since we have a finite collection of sets.

(c) The intersection of any collection of closed subsets is closed:

Let $\{\mathbb{V}(S_\alpha)\}_{\alpha \in I}$ be any collection of closed subsets. We shall show that $\cap_{\alpha \in I} \mathbb{V}(S_\alpha) = \mathbb{V}(\cup_{\alpha \in I} S_\alpha)$.

Indeed, id $x \in \cap_{\alpha \in I} \mathbb{V}(S_\alpha)$, then $f(x) = 0$ for all $f \in \cup_{\alpha \in I} S_\alpha$, thus $\cap_{\alpha \in I} \mathbb{V}(S_\alpha) \subset \mathbb{V}(\cup_{\alpha \in I} S_\alpha)$.

On the other hand, if $f(x) = 0$ for all $f \in \cup_{\alpha \in I} S_\alpha$, then $x \in \mathbb{V}(S_\alpha)$ for every $\alpha \in I$, and so $\mathbb{V}(\cup_{\alpha \in I} S_\alpha) \subset \cap_{\alpha \in I} \mathbb{V}(S_\alpha)$. Therefore, $\cap_{\alpha \in I} \mathbb{V}(S_\alpha) = \mathbb{V}(\cup_{\alpha \in I} S_\alpha)$ and so $\cap_{\alpha \in I} \mathbb{V}(S_\alpha)$ is closed.

(d) Weak separation axiom:

Let $x \neq y$ be in X and define $f_i(z) = z_i - x_i$ for each $i \in [n]$ $(X = \mathbb{F}^n)$, where x_i, z_i denote the i^{th} coordinates of x and z, respectively. Then, $\mathbb{V}(\{f_i\}_{i \in [n]}) = \{x\}$, thus $F = \{x\}$ is a closed subset of X containing x and not containing y.

Proposition 1 *Suppose that* $\mathbb{F}$ *is an infinite field and* $n \geq 1$.

(a) The complement to a hypersurface in $\mathbb{F}^n$ *is an infinite set. In particular the complement to any Zariski closed subset not equal to* $\mathbb{F}^n$ *is an infinite set.*

(b) Every two non-empty Zariski open subsets have non-empty intersection.

(c) If a polynomial $Q(x)$ *vanishes on a non-empty Zariski open subset, then* $P(x) \equiv 0$.

Proof. (a) Perform induction on n.

The base case for $n = 1$ is easy, since any polynomial p has finitely many zeroes, thus $\mathbb{V}(S)$,

$S = \{p\}$, is finite and $\mathbb{V}(S)$ is finite even so more if S contains more than one polynomial. Thus, the complement to any Zariski closed subset not equal to $\mathbb{F}$ is an infinite set.

If $P = P(x_1, x_2, \ldots, x_n) \neq 0$, then write $P = a_0(\bar{x})x_i^N + a_1(\bar{x})x_i^{N-1} + \cdots + a_N(\bar{x})$, where $\bar{x} = (x_1, x_2, \ldots, \hat{x_i}, \ldots, x_N)$ and $a_0(\bar{x}) \neq 0$. By the inductive assumption there are infinitely many points for which $a_0(\bar{x}) \neq 0$ and for each such point there is a value of x_i for which $P(x_1, x_2, \ldots, x_n) \neq 0$. So there are infinitely many points where P does not vanish.

(b) A non-empty Zariski open subset contains the complement to a hypersurface $\mathbb{V}(P)$. Taking two non-empty Zariski open subsets they contain the complements to $\mathbb{V}(P_1)$ and $\mathbb{V}(P_2)$, respectively. Therefore, their intersection contains complement to their union, which is $\mathbb{V}(P_1) \cup \mathbb{V}(P_2) = \mathbb{V}(P_1 P_2)$, and by (a) it contains infinitely many points.

(c) If a polynomial $P \not\equiv 0$ and vanishes on a non-empty Zariski open subset U, then we know that $\mathbb{V}(P)$ is a hypersurface. Furthermore, since the intersection of the complement of $\mathbb{V}(P)$ and U non-empty by (b), we obtain that for x in the intersection of the complement of $\mathbb{V}(P)$ and U $P(x) \neq 0$ and $P(x) = 0$, contradiction.

Regular elements

Let $a \in \mathfrak{g}$, where $\mathfrak{g}$ is a d-dimensional Lie algebra ($d < \infty$) over the field $\mathbb{F}$. Consider the characteristic polynomial of ad a:

$$det_{\mathfrak{g}}(\text{ad } a - \lambda I) = (-\lambda)^d + (tr_{\mathfrak{g}}\text{ad } a)\lambda^{d-1} + \cdots + det_{\mathfrak{g}}\text{ad } a.$$

Note that ad a is a singular operator since (ad a)a=[a, a]=0, hence, $det_{\mathfrak{g}}\text{ad } a = 0$, i.e. the characteristic polynomial of ad a has a vanishing constant term. Write $det(\text{ad } a - \lambda I) = (-\lambda)^d + c_{d-1}(a)\lambda^{d-1} + \cdots + c_r(a)\lambda^r$, where the coefficients $c_{d-1}, c_{d-2}, \ldots, c_0$ are polynomial functions on $\mathfrak{g}$ and r is the smallest integer such that $c_r(a) \not\equiv 0$ (recall that $c_0 \equiv 0$).

Definition 4 *The above r is called the* rank *of $\mathfrak{g}$. An element $a \in \mathfrak{g}$ is called regular if $c_r(a) \neq 0$.*

Proposition 2 *(a) The inequalities $1 \leq r \leq d$ hold, where r is as above, and d is the dimension of the Lie algebra $\mathfrak{g}$.*

(b) The equation $r = d$ holds if and only if $\mathfrak{g}$ is a nilpotent Lie algebra.

(c) If $\mathfrak{g}$ is a nilpotent Lie algebra, then the set of non-regular elements of $\mathfrak{g}$ is $\mathfrak{g}$, whereas if $\mathfrak{g}$ is not nilpotent, then the set of non-regular elements is a complement to a hypersurface in $\mathfrak{g}$. In particular, the set of regular elements is Zariski open, and $\mathfrak{g}$ contains infinitely many regular elements if $\mathbb{F}$ is an infinite field.

Proof. The statement of *(a)* follows since $c_0 \equiv 0$.

In *(b)* $r = d$ means that $det(\text{ad } a - \lambda I) = (-\lambda)^d$, which means that ad a is a nilpotent operator for all a, which is the case if and only if $\mathfrak{g}$ is nilpotent (by Engel's theorem).

(c) If $\mathfrak{g}$ is nilpotent, then $r = d$ and $c_d \equiv 1$, therefore every element of $\mathfrak{g}$ is regular. If $\mathfrak{g}$ is not nilpotent, then we shall use the statement of an exercise that we shall proof later.

Exercise 7.2. The polynomial $c_r(x)$ is homogeneous of degree $d - r$.

Indeed, if $\mathfrak{g}$ is not nilpotent, then $r \neq d$, thus c_r is a non-constant polynomial, and thus the set of non-regular elements of $\mathfrak{g}$ is the hypersurface $\mathbb{V}(c_r(x))$. But, by Proposition 1 the complement to this hypersurface is infinite as $\mathbb{F}$ is infinite.

Solution of Ex.7.2. We shall actually prove the statement for all c_l not only for c_r. Note that the determinant of a matrix $A = (a_{i,j})$ is a homogeneous polynomial in a_{ij}, thus, the determinant of ad $a - \lambda I$ is homogeneous of degree n in $a_{ij}, a_{ii} - \lambda, i \neq j$, where $A =$ ad a. It follows that $det(\text{ad } a - \lambda I)$ is homogeneous in a_{ij} and λ. We are interested in the coefficient of λ^l, and hence of terms that contain exactly l multiples of λ, the rest of the n variables in each term are a_{ij}, so $c_l(x)$ is a homogeneous polynomial of degree $n - i$ in a_{ij}.

Example What are the regular elements of gl_n? Let $\mathfrak{g} = \mathrm{gl}_n(\mathbb{F})$, and $a \in \mathfrak{g}$, $a = a_s + a_n$, where a_s is diagonalizable, a_n is nilpotent, and a_s and a_n commute. Then, ad $a =$ ad $a_s +$ ad a_n, where ad a_s is semisimple, and ad a_n is nilpotent. The answer to the question will be given in the exercise below and in the comments following it, and we shall find that $a \in \mathrm{gl}_n(\mathbb{F})$ is regular if and only if all eigenvalues of the matrix a are distinct.

Exercise 7.3.

(a) If a_s is semisimple with eigenvalues $\lambda_1, \ldots, \lambda_n$, then ad a_s is diagonalizable with eigenvalues $\{\lambda_i - \lambda_j\}$.

(b) ad a has the same eigenvalues as ad a_s.

Solution. (a) Choose a basis of $\mathbb{F}^n$ in which a_s is diagonal, and let e_{ij} be the matrix with zero entry everywhere but the $(i,j)^{th}$ position where it has a 1. Then, ad $(a_s)e_{ij} = a_s e_{ij} - e_{ij} a_s = (\lambda_i - \lambda_j) e_{ij}$, thus ad a_s is diagonalizable with eigenvalues $\lambda_i - \lambda_j$, $i, j \in [n]$.

(b) Take the Jordan decomposition of $a = a_s + a_n$. Then ad a_n is nilpotent since a_n is nilpotent, and by (a) ad a_s is semisimple. Hence, we have a decomposition of ad a into a semisimple and nilpotent part, which commute, thus this decomposition by the uniqueness of the Jordan decomposition is the Jordan decomposition of ad a. Since the eigenvalues of the semisimple part of a Jordan decomposition are the same as those of the original matrix, it follows that ad a and ad a_s have the same eigenvalues.

By exercise 7.3.*(b)* we have that

$det(\text{ad } a - \lambda I) = det(\text{ad } a_s - \lambda I) = \prod_{i,j=1}^{n}((\lambda_i - \lambda_j) - \lambda) =$
$= (-\lambda)^n \prod_{i \neq j}((\lambda_i - \lambda_j) - \lambda)$,

hence $c_j(a) \equiv 0$ for $j = 0, 1, \ldots, n-1$, and $c_n(a) = \prod_{i \neq j}(\lambda_i - \lambda_j) \not\equiv 0$ if and only if the eigenvalues λ_i are all different. Hence, rank $\mathrm{gl}_n(\mathbb{F}) = n$ and $a \in \mathrm{gl}_n(\mathbb{F})$ is regular if and only if all eigenvalues of the matrix a are distinct. The hypersurface of non-regular elements is given by the polynomial

$\prod_{i\neq j}(\lambda_i-\lambda_j)=0$. This polynomial is called the *discriminant.*

Exercise 7.4. Compute explicitly the discriminant for $\mathrm{gl}_2(\mathbb{F})$. Then, find the rank of $\mathrm{sl}_n(\mathbb{F})$.

Solution. The discriminant is $\prod_{i\neq j}(\lambda_i-\lambda_j)=-(\lambda_1-\lambda_2)^2=-(\lambda_1+\lambda_2)^2+4\lambda_1\lambda_2=-(trA)^2+4detA=-(a+d)^2+4(ad-bc)$.

We can find the rank of sl_n in an analogous way as that of gl_n above. Notice, that the only difference is that there is one less zero eigenvalue for ad a_s, that is not hard to see if in the solution of Ex.7.3.(a) one takes the matrices e_{ij} for $i\neq j$ and $e_{jj}-e_{11}$ for $j\neq 1$ (which are all in sl_n) instead of the matrices e_{ij} as we did for gl_n. Thus, we can write $det(\mathrm{ad}\ a-\lambda I)=det(\mathrm{ad}\ a_s-\lambda I)=(-\lambda)^{n-1}\prod_{i\neq j}((\lambda_i-\lambda_j)-\lambda)$,

so $\mathrm{rank}\,\mathrm{sl}_n(\mathbb{F})\geq n-1$. The coefficient of λ^{n-1} is $\prod_{i\neq j}(\lambda_i-\lambda_j)$, which would be identically zero only if all matrices in $\mathrm{sl}_n(\mathbb{F})$ had multiple eigenvalues. This is however not the case. Thus, $\mathrm{rank}\,\mathrm{sl}_n(\mathbb{F})=n-1$.