Lecture 8 – October 5, 2004

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Definition 1. *Let* $\mathfrak{h}$ *be a subalgebra of a Lie algebra* $\mathfrak{g}$*. The* normalizer *of* $\mathfrak{h}$ *in* $\mathfrak{g}$ *is the subalgebra* $N_{\mathfrak{g}}(\mathfrak{h}) = \{a \in \mathfrak{g} | [a, \mathfrak{h}] \in \mathfrak{h}\}$.

It is a subalgebra by the Jacobi identity and one has $\mathfrak{h} \subset N_{\mathfrak{g}}(\mathfrak{h})$.

Lemma 1. *If* $\mathfrak{g}$ *is nilpotent and* $\mathfrak{h} \subsetneq \mathfrak{g}$ *is a subalgebra then* $\mathfrak{h} \subsetneq N_{\mathfrak{g}}(\mathfrak{h})$.

Proof. We have $\mathfrak{g} \supset \mathfrak{g}^2 \supset \ldots \supset \mathfrak{g}^j \supset \mathfrak{g}^{j+1} \supset \ldots \supset \mathfrak{g}^N = 0$ for $N \gg 0$. Consider the maximum j such that $\mathfrak{g}^j \not\subset \mathfrak{h}$, so $\mathfrak{g}^{j+1} \subset \mathfrak{h}$. Then $[\mathfrak{h}, \mathfrak{g}^j] \subset \mathfrak{g}^{j+1}$. Hence $\mathfrak{g}^j \subset N_{\mathfrak{g}}(\mathfrak{h})$. □

Definition 2. *A* Cartan subalgebra *of a Lie algebra* $\mathfrak{g}$ *is a nilpotent subalgebra,* $\mathfrak{h}$ *such that* $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$.

Corollary 1. *Any Cartan subalgebra is maximal among nilpotent subalgebras.*

Exercise 8.1. *The subalgebra* $\mathfrak{n}_n$ *(of strictly upper triangular matrices) is a maximal nilpotent subalgebra in* $\mathfrak{b}_n$*,* $\mathfrak{sl}_n$ *and* $\mathfrak{gl}_n$*, it is not Cartan in any of them.*

Solution. $\mathfrak{n}_n$ is not Cartan since, as was shown earlier, $[\mathfrak{b}_n, \mathfrak{n}_n] \subset \mathfrak{n}_n$ and thus $\mathfrak{b}_n \subset N_{\mathfrak{g}}(\mathfrak{n}_n)$. Unfortunately, $\mathfrak{n}_n$ is not maximal. Consider the algebra $\mathfrak{n}_n + \mathbb{F}I_n \supsetneq \mathfrak{n}_n$. Since I_n commutes with everything, this algebra is equal to $\mathfrak{n}_n \oplus \mathbb{F}I_n$. Hence, it is nilpotent and strictly contains $\mathfrak{n}_n$. □

Correction to Exercise. $\mathfrak{n}_n + \mathbb{F}I_n$ is a maximal nilpotent subalgebra in $\mathfrak{b}_n$ and $\mathfrak{gl}_n$. Unless $n = \text{char}\,\mathbb{F} = 2$, in which the embedding of $\mathfrak{H}_1$ given in Exercise 5.2 is a counter-example. (Note: that $\mathfrak{n}_n + \mathbb{F}I_n$ is not Cartan either since $\mathfrak{b}_n \subset N_{\mathfrak{g}}(\mathfrak{n}_n) \subset N_{\mathfrak{g}}(\mathfrak{n}_n + \mathbb{F}I_n)$.

Solution. We have shown earlier that $\mathfrak{n}_n + \mathbb{F}I_n$ is nilpotent. Note that it suffices to show $\mathfrak{n}_n + \mathbb{F}I_n$ is maximal in $\mathfrak{gl}_n$. Suppose there exists some nilpotent $\mathfrak{h} \supsetneq \mathfrak{n}_n + \mathbb{F}I_n$. First note that in fact we have $\mathfrak{b}_n = N_{\mathfrak{gl}_n}(\mathfrak{n}_n + \mathbb{F}I_n)$, since if $b = \sum_{i,j} c_{ij}E_{ij} \in N_{\mathfrak{g}}(\mathfrak{n}_n + \mathbb{F}I_n)$ with $c_{i'j'} \neq 0$ for $i' > j'$, then $E_{j'i'} \in \mathfrak{n}_n \subset \mathfrak{n}_n + \mathbb{F}I_n$ and $[b, E_{j'i'}] = \sum_i c_{ij'}E_{ii'} - \sum_j c_{i'j}E_{j'j}$. Note that the $(i', i')^{\text{th}}$ and the $(j', j')^{\text{th}}$ entries are $c_{i'j'}$ and $-c_{i'j'}$, both of which are nonzero. Thus $[b, E_{j'i'}] \notin \mathfrak{n}_n + \mathbb{F}I_n$, unless $n = \text{char}\,\mathbb{F} = 2$. Now by the lemma above we must have $\mathfrak{h} \cap N_{\mathfrak{gl}_n}(\mathfrak{n}_n + \mathbb{F}I_n) \supsetneq \mathfrak{n}_n + \mathbb{F}I_n$. Thus $\mathfrak{h}$ contains some element of $\mathfrak{b}_n \setminus \mathfrak{n}_n + \mathbb{F}I_n$, but all elements of $\mathfrak{b}_n \setminus \mathfrak{n}_n + \mathbb{F}I_n$ have at least two distinct eigenvalues, and hence are not ad-nilpotent, contradicting Engel's theorem. □

Examples of Cartan subalgebras:

1. Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then all Cartan subalgebras are just $\mathfrak{g}$.

2. Let $\mathfrak{g} \subset \mathfrak{gl}_n$ be a subalgebra containing a diagonal matrix with distinct eigenvalues. Then $\mathfrak{h} = \mathfrak{g} \cap \{$subalgebra of all diagonal matrices$\}$ is a Cartan subalgebra of $\mathfrak{g}$.

   Proof. $\mathfrak{h}$ is commutative, hence nilpotent. Let $b = \sum_{i,j} c_{ij} E_{ij} \in N_{\mathfrak{g}}(\mathfrak{h})$ then $[a, b]$ is a diagonal matrix, but $[a, E_{ij} = (\lambda_i - \lambda_j)E_{ij}$ where $a = diag(\lambda_1, \ldots, \lambda_n)$. Therefore, $c_{ij} = 0$ for $i \neq j$ and b is a diagonal matrix. □

Proposition 1. *Assume* $\mathbb{F}$ *is an algebraicly closed field of characteristic* 0*. Let* $\mathfrak{h} \subset \mathfrak{g}$ *be a Cartan subalgebra. Since* $\mathfrak{h}$ *is nilpotent, we may consider the generalized root space decomposition:* $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha = \{v \in \mathfrak{g} | (\text{ad } h - \alpha(h))^N v = 0 \text{ for } N \gg 0 \text{ and all } h \in \mathfrak{h}\}$*. Then* $\mathfrak{g}_0 = \mathfrak{h}$*.*

Proof. $\mathfrak{h} \subset \mathfrak{g}_0$ since $(\text{ad } h)^N h' = 0$ for $h, h' \in \mathfrak{h}$ by nilpotency.
Now suppose $\mathfrak{h} \neq \mathfrak{g}_0$. Consider the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}_0$ and $\mathfrak{g}_0/\mathfrak{h}$. Since ad h is a nilpotent operator on $\mathfrak{g}_0$ for each $h \in \mathfrak{h}$, it is also nilpotent on $\mathfrak{g}_0/\mathfrak{h}$. Hence by Engel's theorem, we have $\bar{a} \in \mathfrak{g}_0/\mathfrak{h}$ $\bar{a} \neq 0$, which is annihilated by $\mathfrak{h}$. But this means that if a is a pre-image of $\bar{a}$ in $\mathfrak{g}_0$ then $[\mathfrak{h}, a] \subset \mathfrak{h}$ so $a \in N_{\mathfrak{g}}(\mathfrak{h}) \setminus \mathfrak{h}$, a contradiction. □

Theorem 1. (E. Cartan) *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra over an algebraicly closed field* $\mathbb{F}$*. Let* a *be a regular element of* $\mathfrak{g}$*, and let* $\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{F}} \mathfrak{g}^a_\lambda$ *be the generalized eigenspace decomposition for* ad a*. Then* $\mathfrak{g}^a_0$ *is a Cartan subalgebra of* $\mathfrak{g}$*. Consequently, any finite-dimensional Lie algebra over an algebraicly closed field contains a Cartan subalgebra.*

Proof. The fact that a is regular means that $\dim \mathfrak{g}^a_0 = \text{rank } \mathfrak{g} \leq \dim \mathfrak{g}^b_0$ for all $b \in \mathfrak{g}$.
We can decompose $\mathfrak{g} = \mathfrak{g}^a_0 \oplus V$, where $V = \bigoplus_{\lambda \neq 0} \mathfrak{g}^a_\lambda$. Since $[\mathfrak{g}^a_\lambda, \mathfrak{g}^a_\mu] \subset \mathfrak{g}^a_{\lambda+\mu}$, we see that $[\mathfrak{g}^a_0, V] \subset V$. Hence the adjoint representation induces a representation π of $\mathfrak{g}^a_0$ on V.
First we show that $\mathfrak{g}^a_0$ is a nilpotent Lie algebra. Consider the following subsets of $\mathfrak{g}^a_0$:
$\mathcal{U} = \{u \in \mathfrak{g}^a_0 | (\text{ad } u)_{\mathfrak{g}^a_0} \text{is not a nilpotent operator}\}$,
$\mathcal{V} = \{v \in \mathfrak{g}^a_0 | (\text{ad } v)_V \text{is a nonsingular operator}\}$.
$a \in \mathcal{V}$ by definition of V. Notice that both $\mathcal{U}$ and $\mathcal{V}$ are Zariski open subsets and that $\mathcal{V}$ is not empty. Suppose that $\mathfrak{g}^a_0$ is not a nilpotent Lie algebra. By Engel's theorem this means ad u on $\mathfrak{g}^a_0$ is a non-nilpotent operator for some $u \in \mathfrak{g}^a_0$. This means that $\mathcal{U}$ is non-empty. Thus $\mathcal{U} \cap \mathcal{V} \neq \emptyset$. Take $b \in \mathcal{U} \cap \mathcal{V}$. Then ad b is non-singular on V and not nilpotent on $\mathfrak{g}^a_0$. Hence, $\mathfrak{g}^b_0 \subsetneq \mathfrak{g}^a_0$. This contradicts the regularity of a.
It remains to show that $N_{\mathfrak{g}}(\mathfrak{g}^a_0) = \mathfrak{g}^a_0$. If $b \in N_{\mathfrak{g}}(\mathfrak{g}^a_0)$ then (in particular) $[a, b] \in \mathfrak{g}^a_0$. Therefore, $(\text{ad } a)^N [a, b] = 0$ for $N \gg 0$, thus $(\text{ad } a)^{N+1} b = 0$. Hence $b \in \mathfrak{g}^a_0$, and $N_{\mathfrak{g}}(\mathfrak{g}^a_0) = \mathfrak{g}^a_0$. □

Application. Classification of 3-dimensional Lie algebras over an algebraicly closed field. Let $\mathfrak{g}$ be a 3-dimensional Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra obtained by the above procedure so that $r = \text{rank } \mathfrak{g} = \dim \mathfrak{h}$.

Case 1: rank $\mathfrak{g} = 3$. Then $\mathfrak{g}$ is nilpotent. Hence, either $\mathfrak{g}$ is abelian, or it is non-abelian hence $\dim Z(\mathfrak{g}) = 1$. Therefore, by Exercise 3.1, $\mathfrak{g} \cong \mathfrak{H}_1$, the Heisenberg algebra.

Case 2: rank $\mathfrak{g} = 2$. Then $\mathfrak{h}$ is a 2-dimensional nilpotent algebra. Hence $\mathfrak{h}$ is abelian. Consider its generalized eigenspace decomposition, $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{F}b$.

Exercise 8.2. *Prove that in this case* $\mathfrak{g} \cong \langle a, b | [a, b] = b \rangle \oplus \mathbb{F}c$ $(\mathfrak{h} = \mathbb{F}a + \mathbb{F}c)$.

Solution. Since $\mathfrak{b} \notin \mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$, there must exits some $a \in \mathfrak{h}$ such that $[a, b] \notin \mathfrak{h}$. Thus $[a, b] = b + d$ for $d \in \mathfrak{h}$. Replacing b by $b + d$ we may assume $[a, b] = b$. Since $\dim \mathfrak{h} = 2$, and $\dim \mathfrak{g}/\mathfrak{h} = 1$ there must exist a $c \in \mathfrak{h}$ such that $[b, c] \in \mathfrak{h}$. Then we have: $0 = [b, 0] = [b, [a, c]] = [[b, a], c] + [a, [b, c]] = [-b, c] + 0$, since $\mathfrak{h}$ is abelian. Thus $[b, c] = 0$, and $\mathfrak{g}$ must have the given structure. □

Case 3: rank $\mathfrak{g} = 1$. Then $\mathfrak{h} = \mathbb{F}a$.

Exercise 8.3. *One has* 3 *possibilities:*

1. $[a, b] = b$, $[a, c] = c + b$, $[b, c] = 0$.
2. $[a, b] = b$, $[a, c] = \lambda c$, $[b, c] = 0$,
   where $\lambda \in \mathbb{F} \setminus \{0\}$ *is a parameter.*
3. $[a, b] = b$, $[a, c] = -c$, $[b, c] = a$.

Solution. Let $V = [a, \mathfrak{g}]$. Since $\mathbb{F}a$ is a Cartan subalgebra, V must have dimension precisely 2, and ad a acts on V without singularities. Scaling a if necessary, we may assume one of the eigenvalues of ad $a|_V$ is 1. Suppose first that ad $a|_V$ is not semisimple. Thus ad $a|_V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in some basis, $\{b, c\}$, that is, $[a, b] = b$ and $[a, c] = b + c$. Also $[a, [b, c]] = [[a, b], c] + [b, [a, c]] = [b, c] + [b, b + c] = 2[b, c]$. But 2 can't be an eigenvector of ad a, thus $[b, c] = 0$, this is the first possibility. Now suppose ad $a|_V$ is semisimple. Then ad $a|_V = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ in some basis $\{b, c\}$, that is, $[a, b] = b$ and $[a, c] = \lambda c$. Now $[a, [b, c]] = [[a, b], c] + [b, [a, c]] = [b, c] + [b, \lambda c] = (1 + \lambda)[b, c]$. Thus we must have either $[b, c] = 0$, possibility 2, and in this case λ is arbitrary and uniquely defined by $\mathfrak{g}$ up to to inverting it (swap b and c and scale a appropriately), or $1 + \lambda$ an eigenvalue of ad a, this requires $1 + \lambda = 0$, thus $\lambda = -1$ and that $[b, c]$ be a multiple of a, scaling b if necessary we may assume $[b, c] = a$, possibility 3. □

Exercise 8.4. *Show that the third algebra in case 3 is isomorphic to* $\mathfrak{sl}_2(\mathbb{F})$ (*if the characteristic of* $\mathbb{F}$ *is not* 2), *and that all the algebras from exercise 8.3 are not isomorphic.*

Solution. The standard basis for $\mathfrak{sl}_2(\mathbb{F})$ consists of $\{h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$. With $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$. Setting $a = \frac{h}{2}$, $b = \frac{x}{\sqrt{2}}$, and $c = \frac{y}{\sqrt{2}}$, gives the relations for the third algebra in case 3.

That all the algebras from exercise 8.3 are not isomorphic follows from the proof of exercise 8.3 since every step in the classification, except when otherwise mentioned and dealt with, i.e., that for the second algebra λ and $\frac{1}{\lambda}$ give the same algebra, every step was uniquely determined from the algebra. $\square$