Lecture 9 – October 7, 2004

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Definition. A Polynomial map $f : \mathbb{F}^m \to \mathbb{F}^n$ is a map of the following form,

$$f\begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_m \end{pmatrix} \to \begin{pmatrix} P_1(x_{1\ldots}\ x_m) \\ . \\ . \\ . \\ P_n(x_{1\ldots}\ x_m) \end{pmatrix} \text{ where } P_i \text{ are polynomials.}$$

Exercise 9.1 Let A be a nilpotent operator in a finite dimensional

vector space V over a field $\mathbb{F}$ of charachteristic 0. Let $\operatorname{Exp}(A) = \sum_{j=1}^{\infty} \frac{A^j}{j!}$

Show that $\operatorname{Exp}(A) : V \to V$ is an invertible linear map with inverse $\operatorname{Exp}(-A)$.

Solution. A is nilpotent so $A^m = 0$ for some $m \in \mathbb{N}$.
Let n be the smallest natural number for which $A^n = 0$. Then

$$\operatorname{Exp}(A) = \sum_{j=1}^{n-1} \frac{A^j}{j!}.$$

(1) Exp (A) is linear :

$$(\operatorname{Exp}(A))(c_1 v_1 + c_2 v_2) = \left(\sum_{j=1}^{n-1} \frac{A^j}{j!}\right)(c_1 v_1 + c_2 v_2)$$

$$= \sum_{j=1}^{n-1} \frac{A^j (c_1 v_1 + c_2 v_2)}{j!} = \sum_{j=1}^{n-1} \frac{(c_1 A^j v_1 + c_2 A^j v_2)}{j!}$$

$$= c_1 \sum_{j=1}^{n-1} \frac{A^j v_1}{j!} + c_2 \sum_{j=1}^{n-1} \frac{A^j v_2}{j!} = c_1 \operatorname{Exp}(A) v_1 + c_2 \operatorname{Exp}(A) v_2.$$

(2) $\operatorname{Exp}(-A)$ is the inverse of Exp (A). Note that if we show $\operatorname{Exp}(-A)\operatorname{Exp}(A) = I$,
it follows that $\operatorname{Exp}(A)\operatorname{Exp}(-A) = I$ since $-(-A) = A$

Consider Exp (xA) Exp (cA) and Exp ((x + c) A), where x and c are real
numbers. Since differentiation with respect to x gives the same result on both sides,
and the expressions are equal when $x = 0$, these two expressions are equal
Using $x = 1$ and $a = -1$ gives the desired result.

Exercise 9.2 : If in addition, V is an algebra and A is a nilpotent derivation of V,
then Exp (A) is an automorphism of the algebra.

Solution. We' ve shown in 9.1 that Exp (A) is an invertivble linear map so we only have
left to show that Exp (A) preserves multiplication in V.

First we show by induction that $A^j(v_1 v_2) = \sum_{k=0}^{j} \binom{j}{k} A^k v_1 A^{j-k} v_2$.

Base case : $A(v_1 v_2) = (Av_1) v_2 + v_1 (Av_2)$ by the definition of a derivation

Inductive step : $A^j (v_1 v_2) = A\left(\sum_{k=0}^{j-1} \binom{j-1}{k} A^k v_1 A^{j-k-1} v_2\right)$

$$= \sum_{k=0}^{j-1} \binom{j-1}{k} (A^{k+1} v_1 A^{j-k-1} v_2 + A^k v_1 A^{j-k} v_2) = \sum_{k=0}^{j} \binom{j}{k} A^k v_1 A^{j-k} v_2$$

So $\operatorname{Exp}(A)(v_1 v_2) = \left(\sum_{j=1}^{n-1} \frac{A^j}{j!}\right)(v_1 v_2) = \sum_{j=1}^{n-1} \frac{\sum_{k=0}^{j} \binom{j}{k} A^k v_1 A^{j-k} v_2}{j!}$

$$= \sum_{j=1}^{n-1} \left(\frac{1}{j!}\right) \sum_{i=0}^{n-1} \left(\frac{1}{i!}\right) A^j v_1 A^i v_2 = \left(\sum_{j=1}^{n-1} \left(\frac{1}{j!}\right) A^j v_1\right)\left(\sum_{i=0}^{n-1} \left(\frac{1}{i!}\right) A^i v_2\right)$$

$= (\operatorname{Exp}(A) v_1)(\operatorname{Exp}(A) v_2)$.

Thus Exp (A) is an automorphism of V.

Lemma 3. If $f : \mathbb{F}^n \to \mathbb{F}^n$ is a polynomial map such that $df|_a : \mathbb{F}^n \to \mathbb{F}^n$ is a nonsingular linear operator for some $a \in \mathbb{F}^n$ then $f(\mathbb{F}^n)$ contains a nonempty Zariski open subset containing $f(a)$.

(Note : this is like the implicit function theorem with Zariski open sets replacing open sets)

Lemma 4. Take $\mathfrak{h}$ to be a Cartan subalgebra of $\mathfrak{g}$ with $a \in \mathfrak{h}$ a regular element. Then $\mathfrak{h} \in \mathfrak{g}_0{}^a$.

Proof. $\mathfrak{h}$ is a nilpotent subalgebra so $\operatorname{ad}(a)|_{\mathfrak{h}}$ is nilpotent so $\mathfrak{h} \subset \mathfrak{g}_0{}^a$ But $\mathfrak{g}_0{}^a$ is a nilpotent Lie Algebra and $\mathfrak{h}$ is a maximal nilpotent subalgebra. Thus $\mathfrak{h} = \mathfrak{g}_0{}^a$.

Proof of Theorem. Take $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Then the root space decomposition is given by $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$,
where $\mathfrak{g}_\alpha = \{a \in \mathfrak{g} \mid (\operatorname{ad}(h) - \alpha(h))^n a = 0$ for n sufficiently large$\}$
Moreover, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and $\mathfrak{g}_0 = \mathfrak{h}$.

Note that if $a \in \mathfrak{g}_\alpha$ and $\alpha \neq 0$, then ad (a) is a nilpotent operator (since $(\operatorname{ad}(a))^N \mathfrak{g}_\beta \subset \mathfrak{g}_{\beta+N\alpha}$ and $\mathfrak{g}_\alpha \neq 0$ for a finite number of distinct α' s since the dimension of $\mathfrak{g}$ is finite)
Let $b_1 \ldots b_n$ be the union of bases of $\mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha \neq 0$. Define the polynomial map $f : \mathfrak{g} \to \mathfrak{g}$ by

$$f\left(\sum_{i=1}^{m} x_i b_i + h\right) = \left(\operatorname{Exp}\left(x_1 (\operatorname{ad}(b_1)) \ldots \operatorname{Exp}\left(x_m (\operatorname{ad}(b_m)\right)\right)\right)(h)$$

where $x_i b_i \in \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ for $\alpha \neq 0$ and $h \in \mathfrak{h}$ Note that $\left(\operatorname{Exp}\left(x_1 (\operatorname{ad}(b_1)) \ldots \operatorname{Exp}\left(x_m (\operatorname{ad}(b_m)\right)\right)\right)$ is an automorphism of $\mathfrak{g}$, by Lemma 2. Moreover, f is a polynomial map in the entries h and x_i.

Now, apply Lemma 3. Take $a \in \mathfrak{h}$ such that $\alpha(a) \neq 0$ for all nonzero α for which $\mathfrak{g}_\alpha$ is nonzero.

Compute $df|_a (b + h) = \frac{d}{dt}\Big|_{t=0} \left(f\left(t\left(\sum_{i=1}^{m} x_i b_i + h\right) + a\right)\right)$. Taylor expanding we get

$= ((I + tx_1\ adb_1 + o(t^2)) \ldots (I + tx_m\ adb_m) + o(t^2))(a + th)$

$= \frac{d}{dt}\Big|_{t=0}\ t[b, a] + Ith$

$= \frac{d}{dt}\Big|_{t=0}\ t([b, a] + h)$

$= [b, a] + h$

So $df|_a (b + h) = [b, a] + h$ which is nonsingular since it is the identity on $\mathfrak{g}_0$ and on $\mathfrak{g}_\alpha$
for all nonzero α it is $-$ ad a which is invertible because ad a has the form
$\alpha(a) *$ Identity + nilpotent part and $\alpha(a)$ is nonzero

By Lemma 3, (*) $(\text{Exp}(x_1 (\text{ad}(b_1)) \ldots \text{Exp}(x_m (\text{ad}(b_m)))\mathfrak{h}$ contains a Zariski open subset $\Omega_{\mathfrak{h}}$ of $\mathfrak{g}$, since $x_1 \ldots x_m \in \mathbb{F}$.

Let Ω_r be the set of regular elements of $\mathfrak{g}$. We know it is a nonempty Zariski open set. Let $\Omega_i = \Omega_{\mathfrak{h}_i}$
for $i = 1, 2$.

Since the intersection of finitely many Zariski open sets is nonempty, $\Omega_{\mathfrak{h}_1} \cap \Omega_{\mathfrak{h}_2} \cap \Omega_r$ is nonempty.

Take $b \in \Omega_{\mathfrak{h}_1} \cap \Omega_{\mathfrak{h}_2} \cap \Omega_r$. b is regular and contained in $\sigma_1(\mathfrak{h}_1)$ and in $\sigma_2(\mathfrak{h}_2)$
for some automorphisms σ_1 and σ_2 due to (*).

Hence $\sigma_1^{-1}(b) \in \mathfrak{h}_1$ and $\sigma_2^{-1}(b) \in \mathfrak{h}_2$. These are regular elements in $\mathfrak{h}_1$ and $\mathfrak{h}_2$
respectively, hence by Lemma 4, $\mathfrak{h}_1 = \mathfrak{g}_0^{\sigma_1^{-1}(b)}$, $\mathfrak{h}_2 = \mathfrak{g}_0^{\sigma_2^{-1}(b)}$.

Take $\sigma = \sigma_2^{-1} \circ \sigma_1$. Then $\sigma(\sigma_1^{-1}(b)) = \sigma_2^{-1}(b)$ maps $\mathfrak{h}_1$ to $\mathfrak{h}_2$.

Note : We reduced this theorem to the construction of a certain map. This
idea was developed further by Grothendieck who realized that maps between
objects are often more important than the objects themselves.

Exercise 9.3. Prove the second part of the theorem, ie. that any $\mathfrak{h} = \mathfrak{g}_0{}^a$,
for $\mathbb{F} = \mathbb{C}$, using the implicit function theorem instead of Lemma 3.

Solution. The proof holds as above until the line where Lemma 3 is used. In place of
Lemma 3 we use the implicit function theorem which shows that because $df|_a$
is nonsingular $f(\mathfrak{g})$ contains an open neighborhood $\Omega_{\mathfrak{h}}$ of $f(a)$. We need only
to show that $\Omega_{\mathfrak{h}} \cap \Omega_r$ is nonempty. After this take $b \in \Omega_{\mathfrak{h}} \cap \Omega_r$. Since b is regular
and contained in the image of $\mathfrak{h}$ under some automorphism σ, $\sigma^{-1}(b) \in \mathfrak{h}$ and is a
regular element since it is the image under automorphism of a regular element, and thus $\mathfrak{h} = \mathfrak{g}_0^{\sigma^{-1}(b)}$.

Finally, we show that the intersection of an open set with a nonempty Zariski
open set is nonempty. We use the fact that if a polynomial vanishes on an open
nonempty subset of $\mathbb{C}^n$, the polynomial is identically zero. The corresponding
Zariski open set is the empty set. So if a Zariski open set does not intersect an open
neighborhood in $\mathbb{C}^n$, it is the empty set. The contrapositive of this statement
gives the desired result.

Trace form.

Let π be a representation of a Lie Algebra $\mathfrak{g}$ in a finite dimensional vector space V.

Definition. A trace form on $\mathfrak{g}$ is the following bilinear form : $(a, b)_V = \text{Tr}(\pi(a)\pi(b))$.

Note the following properties of the trace form :
(1) Bilinearity
(2) Symmetry
(3) Invariance (ie. $([a, b], c)_V + (b, [a, c])_V = 0$, which is equivalent to $([a, b], c)_V = (a, [b, c])_V$)

Proof.

(1) Follows from bilinearity of matrix multiplication and the linearity of the trace operation .

(2) Clear, as $\text{Tr}([A, B]) = 0$

(3) $\text{Tr}([\pi(a), \pi(b)]\pi(c)) + \text{Tr}(\pi(b)[\pi(a), \pi(c)])$
$= \text{Tr}(\pi(a)\pi(b)\pi(c) - \pi(b)\pi(a)\pi(c) + \pi(b)\pi(a)\pi(c) - \pi(b)\pi(c)\pi(a))$
$= \text{Tr}([\pi(a), \pi(b)\pi(c)])$
$= 0$
Exchanging a and b gives $([b, a], c)_V + (a, [b, c])_V = 0 \Rightarrow ([a, b], c)_V = (a, [b, c])_V$

Proposition. If $\mathfrak{g}$ is a Lie Algebra and (., .) is an invariant symmetric bilinear form,
then $M^{\perp}$ is an ideal of $\mathfrak{g}$ if M is an ideal of $\mathfrak{g}$. (Where $M^{\perp} = \{a \in \mathfrak{g} \mid (a, M) = 0\}$.)
In particular, $\mathfrak{g}^{\perp} = \ker(., .)$ is an ideal of G.

Proof. If $a \in M^{\perp}$, then $(a, m) = 0$, for any $m \in M$. Then for any $c \in \mathfrak{g}$,
$([a, c], m) = (a, [c, m]) = 0$, since $[c, m] \in M$. Thus for any $a \in M^{\perp}$ and $c \in \mathfrak{g}$,
$[a, c] \in M^{\perp}$ and $M^{\perp}$ is an ideal.