# JUNIOR INDEPENDENT WORK
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This paper represents my own work in accordance with University regulations.

## 1. Introduction

This paper concerns two types of simple graphs:

(1) Graphs possessing the maximum possible girth with a given diameter. We derive a complete characterization of these.
(2) Graphs, which are both their own duals and their own complements. We present a complete listing for ones drawn in the plane and in the torus.

## 2. Maximal-girth Graphs

### 2.1. **Definitions.**

*Notation.* $(u, v)_G$ is the shortest path between $u$ and $v$ with $u, v \in V(G)$ and $(u, v)_G \subseteq E(G)$. $\|(u, v)_G\|$ denotes the length of this path. We will omit the subscript when the context is obvious.

Most of the time, we use paths as sets of edges. However, we will occasionally refer to the path's vertex set; by that, we mean the set of vertices that are endpoints of members of the path. The ends (or endpoints) of a path are the two vertices that have degree 1 in its edge set.

**Definition 2.1.** The girth of G, $g(G)$, is the number of edges in the shortest cycle of $G$. Girth is not defined for forests; therefore, this section excludes them from consideration.

**Definition 2.2.** The diameter of G, $d(G)$ is the maximum of $\|(u, v)_G\|$ over all $u, v \in V(G)$.

**Definition 2.3.** A $d, k$ − Moore graph is a $k$-regular graph of diameter $d$ with $1 + k\sum_{i=0}^{d-1}(k-1)^i$ vertices.

**Definition 2.4.** $G$ is a maximal-girth graph if no graph with the same diameter has a larger girth.

2.2. **Preliminaries.** In this subsection, we present some simple results that will be very useful later. We begin with this trivial lemma:

**Lemma 2.1.** *For any $G$, $g(G) \leq 2d(G) + 1$.*

*Proof.* Suppose there exists G such that $g(G) \geq 2d(G) + 2$. Let $C \subseteq G$ be a cycle with $|V(C)|$ minimum. Let $a, b \in V(C)$ be two vertices separating $C$ into two paths of length at least $d(G) + 1$. If $\|(a, b)_G\| \leq d(G)$, $C$ would not be the shortest cycle of $G$. Hence, $\|(a, b)_G\| \geq d(G) + 1$, which contradicts the definition of the diameter. □

We thus obtain a more precise characterization of maximal-girth graphs:

**Corollary 2.2.** *$G$ is maximal-girth iff $g(G) = 2d(G) + 1$.*

*Proof.* $\Leftarrow$ : Suppose $g(G) = 2d(G) + 1$. By Lemma 2.1, no graph with diameter $d(G)$ and larger girth can exist. Thus, G is maximal-girth.

$\Rightarrow$ : Suppose $G$ is maximal-girth, but $g(G) \neq 2d(G)+1$. Then, from Lemma 2.1, $g(G) < 2d(G) + 1$. Let $G'$ be the cycle of length $2d(G) + 1$. $g(G') > g(G)$, but $d(G') = d(G)$, so $G$ is not maximal-girth. That is a contradiction. □

Another valuable property of maximal-girth graphs is that:

**Lemma 2.3.** *All maximal-girth graphs are 2-vertex-connected.*

*Proof.* Suppose $G$ has a cut-vertex $v$. Then, $G \backslash v$ has more than one component. Assume that some component $G'$ has the property that $v \in C \subseteq (G' \cup \{v\})$, where $C$ is the shortest cycle through $v$. By definition, $C$ has size $\geq g(G)$. Let $u$ be a point in $C$ that is at the distance (in $C$) $l = \left\lfloor \frac{g(G)}{2} \right\rfloor$ from $v$. Since $C$ is the shortest cycle through $v$, $u$ is also $l$ edges away from $v$ in $G$. Now consider $w$ in a component other than $G'$. Since $v$ is a cut-vertex, $(u, v) \subset (u, w)_G$, and $d(G) \geq \|(u, w)_G\| > \left\lfloor \frac{g(G)}{2} \right\rfloor$. Therefore, $g(G) < 2d(G) + 1$ and $G$ is not maximal-girth.

If no component has a cycle containing $v$, we may contract all the edges leading out of $v$ to obtain $G'$. The new graph clearly has the same cycles as $G$, and $v$ is still a cut-vertex. Therefore, $g(G') = g(G)$ and $d(G') \leq d(G)$. We can repeat this procedure indefinitely until $v$ does belong to a cycle in some component, at which point we get the previous case: $g(G) = g(G') < 2d(G') + 1 \leq 2d(G) + 1$. □

2.3. **Main Result.** The following observation motivates our further investigation:

**Proposition 2.4.** *All Moore graphs are maximal-girth.*

*Proof.* Let $G$ be a $d, k$ – Moore graph, and $u \in V(G)$. Label $S_l$ the set of points $v$ such that $\|(u, v)\| = l$. Obviously, $|S_0|$ is 1, and $|S_1|$ is $k$.

In general, for $0 < i \leq d$, every vertex $v \in S_i$ has $k$ edges going to $S_{i-1}$, $S_i$, or $S_{i+1}$. If $v$ had an edge to $S_j$ with $j < i-1$, $\|(u, v)\|$ would be $< i$ — a contradiction since $v \in S_i$. If $v$ had an edge to $w \in S_j$, with $j > i + 1$, $w$ would actually be in $S_{i+1}$ — again, a contradiction.

Obviously, for each $v \in S_i$ at least one of these $k$ edges must go to a $v' \in S_{i-1}$. If $i < d$, the others can (obviously) and must go to distinct vertices in $S_{i+1}$. In other words,

$$|S_{i+1}| = (k - 1)\, |S_i| \tag{1}$$

Suppose some edge other than $(v, v')$ goes to $S_{i-1}$ $(0 < i \leq d)$. Then, the $|S_i| \leq (k-1)\, |S_{i-1}| - 1$, and hence $|V(G)| \leq 1 + k \sum_{j=0}^{i-2} (k-1)^j + (k(k-1)^{i-1} - 1) \sum_{j=0}^{d-i} (k-1)^j < 1 + k \sum_{i=0}^{d-1} (k-1)^i$, so $G$ is not a Moore graph — a contradiction. Similarly, if we suppose that some edge goes from $v$ to $S_i$ $(0 < i < d)$, $|S_{i+2}| \leq (k-1)\, |S_i| - 2$ and $G$ once again cannot be a Moore graph.

Moreover, if some $v_1, v_2 \in S_i$, $0 < i < d$, have a common neighbor $w$ in $S_{i+1}$, then $w$ has two different neighbors in $S_{(i+1)-1}$, which was shown to be impossible above. Therefore, the neighbors of every $v_1 \in S_i$ in $S_{i+1}$ are distinct from those of every other $v_2 \in S_i$. That proves (1).

Therefore, every cycle through $u$ is of length at least $2d+1$ (otherwise, (1) would have to be violated). Since $u$ was chosen arbitrarily, $g(G) = 2d + 1$. □

This proposition makes it natural to ask whether there are any non-Moore maximal-girth graphs. Arriving at the answer becomes a relatively simple matter once we describe Moore graphs as follows:

**Lemma 2.5.** *A regular maximal-girth graph is a Moore graph.*

*Proof.* Suppose $G$ is a $k$-regular maximal-girth graph. Using the notation from the proof of Proposition 2.4, we want to show that $\forall 0 < i < d(G)$, $|S_{i+1}| = (k-1)\,|S_i|$. As in Proposition 2.4, given $v \in S_i$ with $0 < i \le d(G)$, the $k$ edges leaving it go to $S_{i-1}$, $S_i$, or $S_{i+1}$, and at least one, $(v, v')$, goes to $S_{i-1}$.

Suppose some $(v, w)$, $v' \neq w \in S_{i-1}$ is an edge. Let $C' = (u, v') \cup (u, w) \cup (w, v) \cup (v', v)$. It is the union of two paths, each at most $d(G)$ in length. The paths have the same endpoints, but do not share all vertices. Therefore, $C'$ contains the edge set of a cycle. The cycle must be at most $2d(G)$ in length — a contradiction, since $G$ is maximal-girth.

Suppose $i < d(G)$ and some $(v, w)$, $w \in S_i$ is an edge. As before, let $C' = (u, v) \cup (u, w) \cup (v, w)$. $C'$ then contains the edge set of a cycle, and is of size $< 2d(G) + 1$ — again, a contradiction.

Now we know that $k - 1$ neighbors of every $v_1 \in S_i$ are in $S_{i+1}$; these neighbors are also distinct from those of every other $v_2 \in S_i$. If that was not the case, the shared neighbor $w \in S_{i+1}$ would have two different neighbors in $S_{(i+1)-1}$, previously shown to be impossible.

Therefore, $|S_{i+1}| = (k-1)\,|S_i|$ and $|V(G)| = \sum_{i=0}^{d(G)} |S_i| = 1 + k \sum_{i=0}^{d-1} (k-1)^i$. Hence, $G$ is a Moore graph. □

Now we can rephrase our question as whether there are any non-regular maximal-girth graphs.

**Proposition 2.6.** *All maximal-girth graphs are regular.*

*Proof.* Let $G$ be maximal-girth with $d(G) = n$, and $C$ — some cycle of size $g(G) = 2n + 1$ with vertices $v_0, v_1, \ldots, v_{2n}$ (subscripts in $\mathbb{Z}/(2n+1)$; such a cycle exists by definition of $g(G)$). Suppose some $v = v_i$ has $\deg v = d + 2$ neighbors: $v_{i+1}, v_{i-1}, v_i^{(1)}, ..., v_i^{(d)}$. No $v_i^{(j)}$ is in $V(C)$: if it were otherwise, there would exist a cycle shorter than $g(G)$ (contradicting the definition of girth).

Consider $u = v_{i-n}$ and some $v' = v_i^{(j)}$. Define $C'$ with $V(C') = V(C) \cup \{v'\}$ and $E(C') = E(C) \cup \{(v, v')\}$. There are three paths of interest: $p_C = (u, v')_{C'}$, $p_{\bar{C}} = (E(C) - (u, v)) \cup (v, v')$, and $p_G = (u, v')_G$.

$\|p_C\| = n+1$, while $\|p_G\| \le n$. Thus, these two paths are not identical. Moreover, it cannot be that $V(p_G) \subset V(p_C)$; if that were the case, by substituting $p_C$ in $C$ with $p_G$ we would obtain a shorter cycle — a contradiction. Hence, we have two paths with common ends that do not share all vertices. Therefore, their union contains a cycle of size at most the sum of their lengths, in our case $\le (n+1) + n$.

Equality must hold, since smaller cycles are impossible. Thus,

$$V(p_C) \cap V(p_G) = \{u, v'\} \text{ and } \|p_G\| = n \tag{2}$$

Also we get, $\|p_{\bar{C}}\| = n + 2$; reasoning as above, the union of these two paths must contain a cycle of length $\leq 2n + 2$. Again, equality must hold, since the next smallest possible value is $2n$ — if any vertices, but the endpoints, are shared between the two paths, the cycle cannot exceed that size. Therefore, once more, $V(p_{\bar{C}}) \cap V(p_G) = \{u, v'\}$.

We thus see that $p_G$ overlaps with $V(C)$ in only one vertex — $u$. Thus, $u$ must have a neighbor aside from $v_{i-n-1}$ and $v_{i-n+1}$ corresponding to $v' = v_i^{(j)}$. Call this neighbor $u^{(j)}$.

Now we must show that for any $v'' = v_i^{(k)}$, such that $k \neq j$, $V((u, v'')) \cap V((u, v')) = \{u\}$. Let $p' = (u, v') \cup (v', v)$ and $p'' = (u, v'') \cup (v'', v)$; then, by (2), $\|p'\| = \|p''\| = n + 1$. As before, the paths share endpoints, but are partially disjoint (since $v' \neq v''$), so $V(p') \cup V(p'')$ contains a cycle of length at most $2n + 2$. Again, if $(V(p') \cap V(p'')) \triangle \{u, v\} \neq \emptyset$, the cycle is at most $2n$ in size, an impossibility. Thus, $u^{(j)} \neq u^{(k)}$ whenever $j \neq k$, and so $\deg u \geq d + 2$.

Therefore, for any $0 \leq i \leq 2n \in \mathbb{Z}/(2n + 1)$, $\deg v_{i-n} \geq \deg v_i$. Thus, $v_{i+1} = v_{i-2d} \geq v_i$ and hence all degrees in the cycle are the same.

Notice that (2) also implies that $v_k^{(j)}$ is in a cycle of size $g(G)$ containing $v$. Therefore, every neighbor of a vertex of $C$ also has degree $d+2$. Analogously, every neighbor of those neighbors is of the same degree. Continuing in this manner, every vertex connected to $v$ is of degree $d + 2$. Since the graph is connected (by Lemma 2.3), it is regular. □

Putting Proposition 2.4, Lemma 2.5, Proposition 2.6 together, we trivially get our final result:

**Corollary 2.7.** *The maximal-girth graphs are exactly the Moore graphs.*

## 3. Self-dual, Self-complementary Graphs

3.1. **Possible Embeddings.** Evidently, in any embedding, a self-dual graph has as many faces as it does vertices. Since we are interested only in simple graphs, a self-dual graph must have all vertices of degree at least 3 — a degree 1 vertex becomes a loop in the dual, while a degree two vertex leads to a double edge. On the other hand, any self-complementary graph has exactly half the possible number of edges. Finally, a graph drawn on a surface of genus $g$ has the following relationship between the number of vertices – $v$, faces – $f$, and edges – $e$: $v - e + f = \chi(g)$, where $\chi(G) = 2 - 2g$ is a function of the genus. (This is the generalized Euler formula, also known as the Poincaré formula, see [1]). These three relationships constrain the possible graphs, and their possible embeddings a great deal — there is no need to consider a graph or a surface with properties that do not yield positive integer (integer for $\chi(g)$ — *Is this true?*) solutions to the following system of equations:

$$\begin{aligned} v - e + f &= \chi(g) \\ v &= f \\ e &= \frac{v(v-1)}{4} \end{aligned} \tag{3}$$

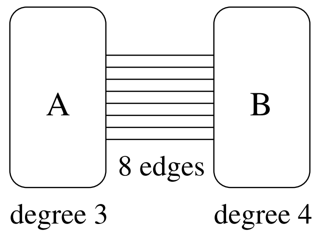


FIGURE 1. The form of 8-vertex self-dual, self-complementary planar graphs. $|A| = |B| = 4$; $A = \bar{B}$.

Simplifying, we get $v^2 - 9v + 4\chi(g) = 0$; solving for $v$,

$$v = \frac{9 \pm \sqrt{81 - 16\chi(g)}}{2} \tag{4}$$

Evidently, the discriminant $D$ is always odd, so whenever it is a perfect square, $v$ is an integer. $D$ takes on the same positive values as $16b+1$ with $b \geq 0$; therefore, we are interested in $(a, b)$ such that $a^2 = 16b + 1$. This is easily shown to happen for values of $a \in \{a_i\}$, where $a_1 = 1$, $a_{2n} = a_{2n-1} + 6$, $a_{2n+1} = a_{2n} + 2$. The corresponding values of $\chi(g) = \frac{81-a_i^2}{16}$; the first few are 5, 2, 0, $-9$, $-13$, $-28$, ... For $\chi(g) = 5$ and 2 both values of $v$ are positive: $4, 5$ and $1, 8$. For the rest, only $\frac{a_i+9}{2}$ is positive. The first few values of that sequence are 5, 8, 9, 12, 13, 16, ... These values are all obviously $\equiv 1 \pmod 4$) or $\equiv 0 \pmod 4$, so $\frac{v(v-1)}{4}$ is always an integer. The corresponding values of the genus are $-\frac{5}{2}$, 0, 1, $\frac{11}{2}$, $\frac{13}{2}$, 15, ... The first one is obviously nonsensical, while for the last 3 listed values the surfaces are complex, and the graphs have to be rather large, making working with them more difficult. Values would beyond clearly yield even more unmanageable cases. Therefore, we only present complete listings of the self-dual, self-complementary graphs on the surfaces of genus 0 and 1 — the plane and the torus.

3.2. **Planar Embedding.** In this case $\sqrt{D} = 7$ and we have two choices for $v = 1, 8$. The graph on one vertex is obvious. As for the graph on 8 vertices, only degrees 3 and 4 are allowed — vertices of degree $\geq 5$ would force vertices of degree $< 3$ in the complement. Moreover, there are exactly 4 of both kinds of vertices, because taking the complement replaces the degree 4 vertices by those of degree 3 and vice versa. Self-complementarity also requires the subgraphs of degree 3 and 4 to be complements, and exactly half of the possible edges ($\frac{4 \cdot 4}{2} = 8$ of them) between these subgraphs to be present. Figure 1 summarizes our findings so far.

$\sum_{v \in A} \deg v = 12$, and 8 of these must go to $B$, so there are exactly two internal edges in $A$. Similarly, there are 4 internal edges in $B$. That leaves only two possible configurations of $A$ and $B$, shown in Figure 2.

Consider configuration 1 — vertex $A_2$ has 3 edges going to $B$, which means it misses one vertex: $B_1$, $B_2$ or $B_3$ ($B_4$ is perfectly symmetrical to $B_3$). The two ways of drawing $B$ in the plane (placing $B_1$ inside or outside the triangle) are equivalent, so we will use the one from the figure. Then, we have 4 cases: if $A_2$ misses $B_1$, we may place it either inside, or outside the triangle — two cases, if $B_2$ or $B_3$ is missed, $A_2$ must necessarily go outside the triangle, giving another two cases. Further analysis of the cases is presented below. The results are in Figure 3.

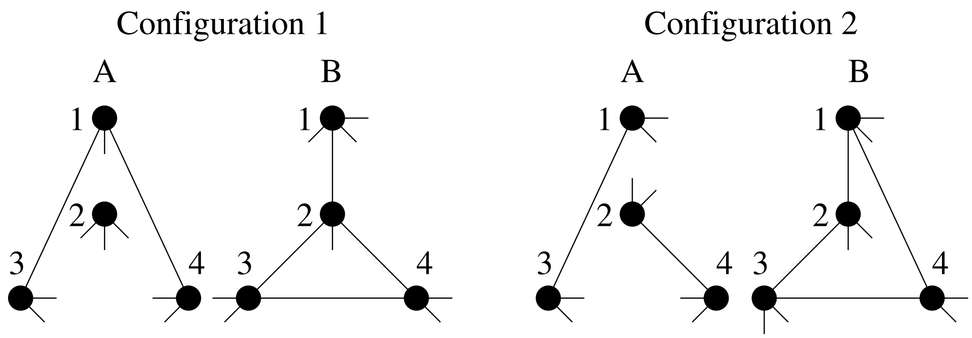


FIGURE 2. Allowed internal configurations of $A$ and $B$.

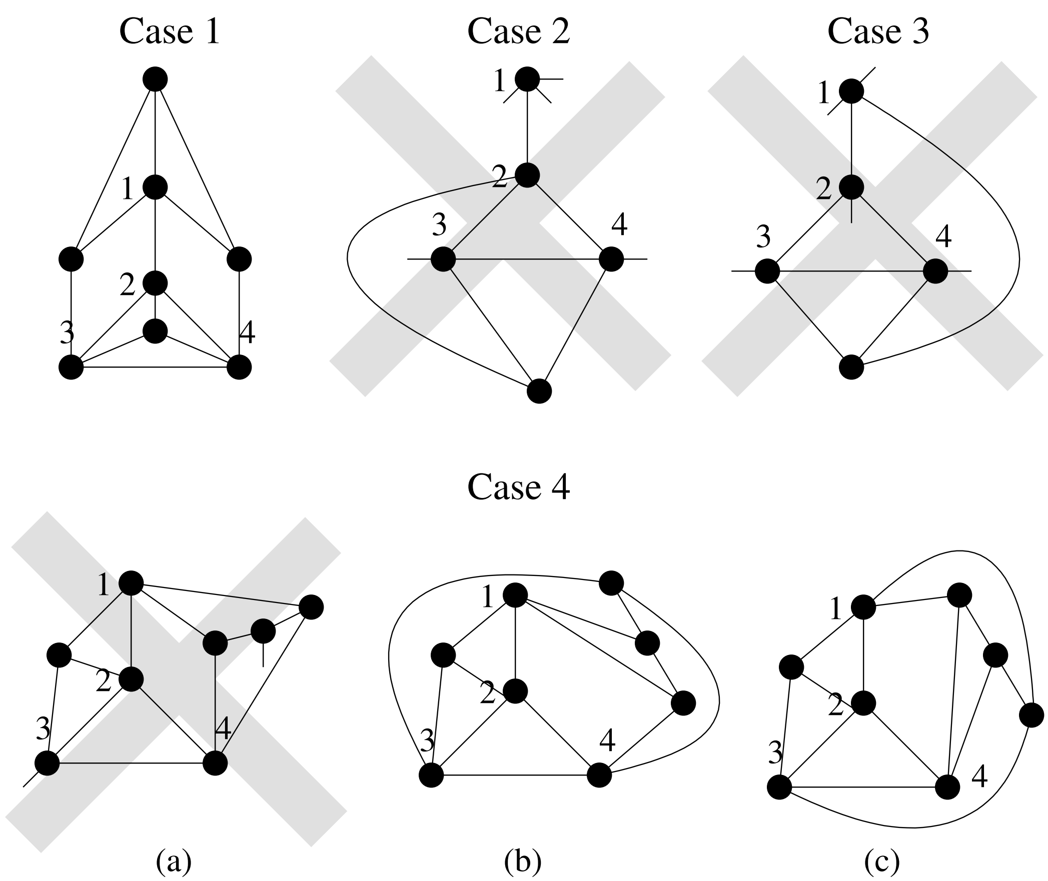


FIGURE 3. Cases occurring in the analysis of configuration 1. Structures that cannot be made planar are crossed out in gray.

(1) If $A_2$ is inside the triangle, $B_1$ has 3 available edges, which must go to $A_1$, $A_3$, and $A_4$. That leaves one available edge on each of $A_3$, $A_4$, $B_3$, $B_4$. The two ways of connecting them produce the same drawing of the same graph, shown in Figure 3. The graph has 5 faces of size 3, so it is not self-dual.

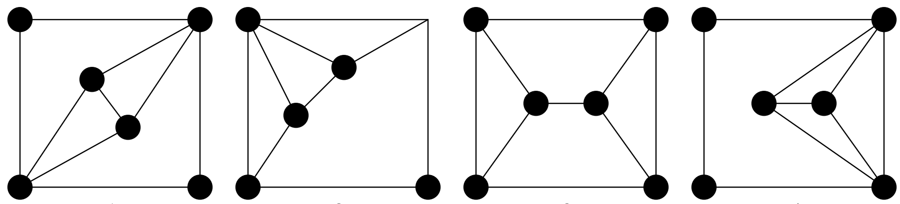


FIGURE 4. Cases occurring in the analysis of configuration 2.

(2) If $A_2$ is outside of the triangle (and misses $B_1$), three vertices of $B$ are left with unconnected edges — $B_1$, $B_3$, and $B_4$. Since the remaining vertices of $A$ are all connected, the three vertices of $B$ must be in a the same face for us to make a planar drawing. However, by inspection, this is impossible (no matter how we draw the connections from $A_2$). The figure shows one way to place the connections (the others are equivalent, with the outside face swapped with some other one).
(3) If $A_2$ misses $B_2$, all four vertices of $B$ retain available edges after $A_2$ is attached; however, it is clear by inspection that they will not be in the same face regardless of the placement of $A_2$. That shows no such graphs are possible by an argument analogous to that of case 2.
(4) If $A_2$ misses $B_3$, the three vertices of $B$ with available edges lie in the same face. Therefore, we may attach the three remaining vertices of $A$. If both $B_4$ and $B_1$ (with two free edges) miss $A_1$, it becomes inaccessible from $B_3$ (case (a)). Thus, the only available configurations are with either $B_1$ or $B_4$ hitting $A_1$. That gives cases (b) and (c), respectively (only one drawing possible in each case). Case (b) has a face of size 5, and so is not self-dual. Case (c) is not self-dual because every face of size 3 has another face of size 3 adjacent (configuration 1 graphs don't have this property).

We thus see that no configuration 1 graph can be self-dual and self-complementary.

In configuration 2 graphs, the inside and the outside of $B$ are perfectly symmetrical, as are the two components of $A$. Therefore, there are only a few cases. By symmetry, we may assume that one component of $A$ called $A'$ is inside $B$ ($A''$ is the other one). It contacts either two, three, or four of $B$'s vertices. There are two ways it can contact 2 vertices, and one way it can contact 3 or 4 vertices, as shown in Figure 4. Case 4 can be ruled out immediately — it has three size 3 faces all adjacent, and none of these can be split up by the introduction of $A''$. As for the others, more detailed analyses follow.

(1) $A''$ cannot be placed inside the square without violating planarity, and must thus go outside. Moreover, there is only one obvious way of connecting it up. The resulting graph is self-complementary and self-dual and is shown in Figure 5.
(2) Again, there is just one way of connecting up $A''$. However, it may now be placed inside or outside $B$. Positioning it inside is excluded as it creates a face of size 6. The remaining choice is to place it ouside; the resulting graph also satisfies the desired conditions, and is in the figure.
(3) Clearly, $A''$ cannot be placed inside $B$; however, there are two distinct ways of connecting it. One yields a self-complementary, self-dual graph, while

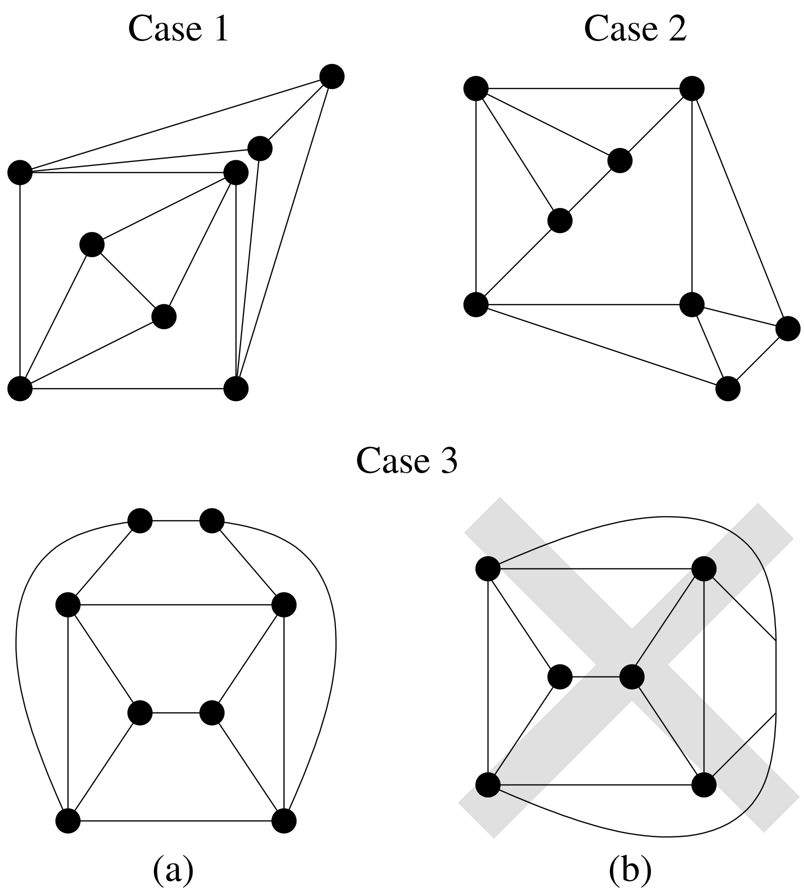


FIGURE 5. The 8-vertex self-dual, self-complementary graphs (1, 2, 3a) and a failing candidate, crossed out in gray (3b).

the other does not. Both are in Figure 5; the unsuitable one is crossed out in gray (b). It is not self-dual, since its size 3 faces are all disconnected.

Thus, there are four self-dual, self-complementary graphs in the plane — the single vertex, and the three 8-vertex graphs listed in Figure 5.

3.3. **Toroidal Embedding.** In this case, self-dual, self-complementary graphs must all have 9 vertices. That also implies, by the same logic as in the previous subsection, that these vertices are all of degree 3, 4, and 5. The numbers of degree 3 and degree 5 vertices are equal, and the restrictions to those vertices are complementary subgraphs, with exactly half the intervening edges present. That excludes graphs with an odd number of degree 3 vertices, leaving three cases: 9 degree 4 vertices; 5 degree 4, 2 degree 3, 2 degree 5; 1 degree 4, 4 degree 3, 4 degree 5. From here, it is possible to continue the same kind of configuration-based analysis as before. However, that is tedious and more difficult, since there are many more cases. Additionally, toroidal embeddings are much less restrictive, and often not unique for a given graph.

The following observations enable relatively simple and efficient computer-based analysis of the problem. The complement operation obviously defines a permutation $p : V(G) \to V(G)$ on a self-complementary graph. If $(u, v) \in E(G)$, $(p(u), p(v)) \notin E(G)$; similarly, if $(u, v) \notin E(G)$, $(p(u), p(v)) \in E(G)$. We will call the graphs that have this property *complement-invariant under p*.

Suppose our permutation has a cycle $(a_1\ a_2\ a_3\ \ldots\ a_n)$ such that $n$ is odd. Let $e((a_i, a_j))$ be 1 if $(a_i, a_j)$ is an edge, and 0 otherwise. Then, if $v = e((a_1, a_2))$ and $w = (v + 1) \mod 2$, $e((a_2, a_3)) = w$, $e((a_3, a_4)) = v$, and so forth, with $e((a_{2i}, a_{2i+1})) = w$ and $e((a_{2i-1}, a_{2i})) = v$ for $1 < 2i < n$. Then, $e((a_{n-1}, a_n)) = w$, $e((a_n, a_1)) = v$, and $e((a_1, a_2)) = w$, a contradiction. Therefore, the complement permutation has no odd cycles. Similarly, it cannot have a two-cycle $(a_1\ a_2)$ — then $e((a_1, a_2)) = v \neq w = e((a_2, a_1))$. It also clearly cannot have more than one fixed point. Therefore, there are only two possibilities for a permutation on 9 vertices — one with two 4-cycles and a fixed point, and one with an 8-cycle and a fixed point. We may label the vertices so that the permutation is either $p_1 = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8)(9)$ or $p_2 = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)(9)$.

Complement-invariance under $p_1$ or $p_2$ constrains the possible graphs greatly. For instance, if we choose $(1, 2)$ to be an edge under $p_1$, $(2, 3)$ and $(4, 1)$ are forced to be non-edges, and $(3, 4)$ is forced to be an edge. For $p_1$, there is a total of 10 such choices to make, instead of 18 for a complete search of 18-edge graphs. $2^{10} = 1024$ graphs is a much more manageable quantity than $\binom{36}{18} = 326704870800$. The constraints on the allowed degrees reduce both numbers substantially. Nonetheless, even after eliminating the graphs with obviously unsuitable degrees from the $\binom{36}{18}$ cases, the brute force approach is problematic for a computer, because checking for graph isomorphism is an expensive operation, and far too many such checks are still necessary.

For $p_2$ there are just five choices, giving 32 more possible graphs, all of which turn out to be included in the 1024 generated by $p_1$. However, it is much faster to prove by calculation than by argument, so we make no further comment on the issue.

Of these 1056 graphs, most are isomorphic; so many, in fact, that it is perfectly practical to make a pass through the graphs taking out one isomorphism class after another. The whole process only takes several tens of thousands of isomorphism checks, yielding 36 distinct graphs. Each of these is self-complementary by construction, so it remains to check if any are self-dual. Of them, 22 have the proper degree structure. That is still too many to comfortably treat manually.

To check for self-duality, we enumerate all possible rotation systems on each graph, generate the dual dictated by the rotation system (walking around each face by turning right at every vertex, as described in section 1.2 of [2]). This method is inefficient, but is simple and fast enough for the small number of small graphs that we have. Depending on the number of degrees, the numbers of rotation systems are as follows: with 4 degree 5 vertices we have $4!^5 3!^2 2!^5 = 31850496$, with 2 degree 5 vertices, $4!^2 3!^5 2!^2 = 17915904$, and, finally, for 9 degree 4 vertices, we have $3!^9 = 10077696$ rotation systems.

We immediately discard every dual with a face larger than the maximum (smaller than the minimum) degree of the tested graph. In most cases, that leaves only a few hundred duals out of the several tens of millions. These duals are then compared for isomorphism with the original. The results — four self-dual, self-complementary

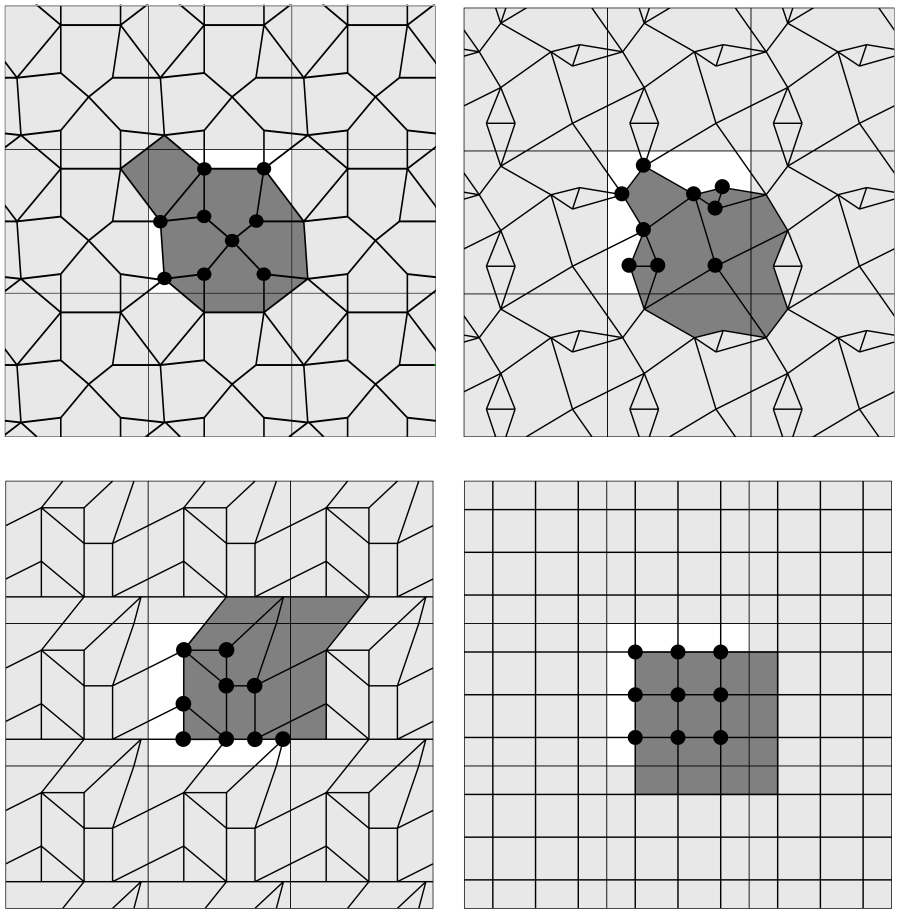

FIGURE 6. The self-complementary, self-dual graphs on the torus.

graphs on the torus are presented in Figure 6, drawn in a manner similar to the one used in [2].

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[2] A. Gagarin, W. Kocay and D. Nelson *Embeddings of Small Graphs on the Torus*, June 2001, CUBO, to appear, available at `http://www.inf.fu-berlin.de/~rote/Lere/2002-SS/Seminar%20Graphenalgorithmen/Kocay-Torus-Embeddings.pdf`