PATTERN-AVOIDANCE IN BINARY FILLINGS OF GRID SHAPES

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Abstract.

A *grid shape* is a set of boxes chosen from a square grid; any Young diagram is an example. This paper considers a notion of pattern-avoidance for 0-1 fillings of grid shapes, which generalizes permutation pattern-avoidance. A filling avoids some patterns if none of its sub-shapes equal any of the patterns. We focus on patterns that are *pairs* of 2×2 fillings. For some shapes, fillings that avoid specific 2×2 pairs are in bijection with totally nonnegative Grassmann cells, or with acyclic orientations of bipartite graphs. We prove a number of results analogous to Wilf-equivalence for these objects — that is, we show that for certain classes of shapes, some pattern-avoiding fillings are equinumerous with others.

Rsum Une *forme de grille* est un ensemble de cases choisies dans une grille carre; un diagramme de Young en est un exemple. Cet article considre une notion de motif exclu pour un remplissage d'une forme de grille par des 0 et des 1, qui gnralise la notion correspondante pour les permutations. Un remplissage vite certains motifs si aucune de ses sous-formes n'est gale un motif. Nous nous concentrons sur les motifs qui sont des *paires* de remplissages de taille 2×2. Pour certaines formes, les remplissages vitant certaines paires de taille 2×2 sont en bijection avec les cellules de Grassmann totalement positives, ou bien avec les orientations acycliques de graphes bipartis. Nous dmontrons plusieurs rsultats analogues l'quivalence de Wilf pour ces objets — c'est--dire, nous montrons que, pour certaines classes de formes, des remplissages vitant un motif donn sont en nombre gal d'autres remplissages.

Revision History

November 25, 2007: Current revision.
April-May, 2008: Upcoming revision. It will incorporate a proof of the $\left(\begin{smallmatrix}\circ & \bullet \\ \circ & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \bullet \\ \circ & \circ\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}\circ & \circ \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \bullet & \circ\end{smallmatrix}\right)$ bijection (and symmetries). The equivalences $\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \bullet \\ \bullet & \circ\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \bullet \\ \bullet & \circ\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}\circ & \circ \\ \circ & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \circ & \circ\end{smallmatrix}\right)$ will have much better descriptions. The motivation and structure of the paper will be improved.

1. Introduction

1.1. **Pattern-avoidance of fillings, in a nutshell.** Perhaps the best-known example of pattern avoidance is defined for permutations. Let S_n be the set of permutations of $[n] = \{1, 2, \ldots, n\}$. A permutation $\sigma \in S_n$ *avoids* $\tau \in S_k$ if there is no set of indices $1 \leq i_1 \leq \cdots < i_k \leq n$ such that

$$\pi(i_{\tau^{-1}(1)}) < \pi(i_{\tau^{-1}(2)}) < \cdots < \pi(i_{\tau^{-1}(n)}).$$

Key words and phrases. pattern-avoidance, filling, grid shape, Le-diagram, acyclic orientation, Young diagram.

In other words, if we take the jth largest value of $\{\pi(i_l)\}$ and replace it by j in $\pi(i_1), \pi(i_2), \ldots, \pi(i_k)$ for all $1 \leq j \leq k$, we will not get τ's word $\tau(1), \tau(2), \ldots, \tau(k)$ for any index set $\{i_l\}$.

Let $S_n(\sigma)$ be the set of permutations of $[n]$ that avoid σ. Permutations σ and τ are *Wilf-equivalent* if $|S_n(\sigma)| = |S_n(\tau)|$ for all n. For more details on Wilf equivalence, and further references, see [8].

The permutation matrix of σ is a matrix $P_\sigma = (p_{ij})$ with $p_{ij} = 1$ if $\sigma(i) = j$, and $p_{ij} = 0$ otherwise. In terms of these matrices, the permutation $\sigma \in S_n$ avoids $\tau \in S_k$ if no $k \times k$ minor of P_σ equals P_τ. If we draw the permutation matrices with lines separating rows and columns, then permutations are just special cases of $0-1$ fillings of square shapes.

Thus, pattern-avoidance generalizes naturally to fillings of shapes, as follows: a filling F avoids a filling G if no minor of F equals G (*both* the shapes and fillings must agree). Wilf-equivalence also translates to this context — two patterns p_1 and p_2 are *equivalent* if p_1-avoiding fillings are equinumerous with p_2-avoiding fillings.

1.2. **Other notions of pattern-avoidance in fillings.** We will use the above definition of pattern-avoiding fillings. However, this is a recent subject with a lot of variation in the objects of study and definitions. Here, we give an overview of some of these variations.

In [5], Marcus and Tardos generalize permutation-avoidance in a different way. In that paper, a filling *contains* a pattern not only if the some minor equals the pattern, but also if the minor has a 1 wherever the pattern has a 1. This definition leads to nice extremal results in permutation-pattern avoidance, but does not seem related to ours.

In [4], Christian Krattenthaler discusses the same object — binary fillings of grid shapes, — but with a rather different definition of "pattern." The main objects in his paper are various chains in fillings. For example, an *NE-chain* is a sequence of 1s in the filling, so that each 1 is above and to the right of the previous one. The chain's length is just the number of 1s in it. Many variations on the notion of chain appear in the paper, and the results describe fillings of Ferrers shapes (Young diagrams, reflected) with restricted chain lengths. These include bijections with other objects, and some statistics. In particular, results on non-crossing set partitions and matchings follow from the results on fillings.

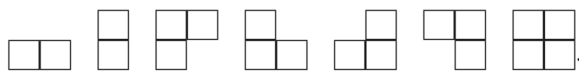
Although the problems in Krattenthaler's paper are quite different from ours, there are some interesting commonalities. Our paper, like his, uses $0-1$ fillings to extend previous results. Another benefit is a more uniform approach to a class of problems: both papers can potentially bring out combinatorial connections that would otherwise be obscured. Finally, Krattenthaler shows interest in more general classes of shapes, something that is at the heart of the present paper.

In [6], Anna de Mier uses a definition of avoidance close to that of Marcus and Tardos [5], but obtains results on noncrossing and nonnesting graphs, related to those in Krattenthaler's paper.

1.3. **Related results.** In [8], Zvezdelina Stankova does not explicitly mention pattern-avoiding fillings. However, the main relation of that paper is shape-Wilf-ordering $\preceq_s$, which can be rephrased in terms of fillings. Let τ and σ be permutations; then $\sigma \preceq_s \tau$ iff for every Young diagram λ, the number of $0-1$ fillings with exactly one 1 in each row and column, which avoid P_σ, is at most the number of such fillings avoiding P_τ. The key point here is that, just as in our definition, every cell of the square pattern P_σ must be inside the Young diagram to match.

In [1], Vt Jelnek studies a problem related to Stankova's. He works with $0-1$ fillings of rectangular shapes. However, instead of constraining all row and column sums to be 1 (that would make a permutation, of course), he allows an arbitrary fixed sum for each row and column. In our terms, his main result is about equivalent patterns (recall — that means "fillings avoiding them are equinumerous"). He shows that permutations of a fixed order ≤ 3 are all equivalent, when restricted to fillings with a fixed multiset of row and column sums.

The results most closely related to ours are due to Kitaev, Mansour, and Vella [3], and Kitaev [2]. In both papers, the shapes are rectangles, and the fillings are binary. They consider all nontrivial patterns up to size 2×2 — that is, all $0-1$ fillings of these shapes:

(1) 

The first paper counts, for each of the 56 described patterns, the number of fillings of an $m \times n$ rectangle, which avoid it. It defines two notions we also use: pattern complement (see Fact 2.12), and pattern symmetry (Subsection 3.1).

The second paper finds equivalences between patterns consisting of multiple 3-cell fillings, as in (1). They forbid 2, 3, and 4 patterns simultaneously, and give equivalence classes of tuples of 3-cell fillings in each case. The main approach, like in [2], is to explicitly count the pattern-avoiding fillings of rectangles.

1.4. **Our results.** Permutations are special fillings of special shapes; all generalizations in the above papers are restricted to special shapes, and often special fillings too. In contrast, in this paper we study arbitrary fillings of arbitrary shapes. However, our patterns are quite special — we require that a filling avoid a *pair* of 2×2-fillings. We made this choice because of some results due to Alex Postnikov [7]. He proved the following facts, which we restate in terms of 2×2 pattern pairs:

(1) Fillings of Young diagrams λ that avoid the pattern pair $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \bullet\\ \bullet & \circ\end{smallmatrix}\right)$ (where $\circ = 0$ and $\bullet = 1$) are exactly the ⅃-diagrams. The latter are in bijection with totally nonnegative Grassmann cells, which are defined in [7] as elements of a particular cellular decomposition of $Gr_{k,n}^{tnn}$. This $Gr_{k,n}^{tnn}$ denotes, in turn, elements of the Grassmannian with nonnegative Plcker coordinates.
(2) Fillings of Young diagrams λ that avoid $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ\\ \circ & \bullet\end{smallmatrix}\right)$ are acyclic orientations of the diagram's bipartite graph G_λ (rows and columns are vertices, boxes are edges). According to [7], the following objects associated with λ are equinumerous: acyclic orientations of G_λ, ⅃-diagrams on λ, totally nonnegative Grassmann cells inside the Schubert cell Ω_λ, and several others.

Thus, $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \bullet\\ \bullet & \circ\end{smallmatrix}\right)$- and $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ\\ \circ & \bullet\end{smallmatrix}\right)$-avoiding fillings are equivalent in Young diagrams.

This paper makes some steps towards a classification of equivalences of 2×2 pattern pairs for general shapes and fillings. The essential notations and definitions are in Section 2, a general discussion of the problem is in Section 3. In Section 4, we extend Postnikov's proof of the above equivalence to far more general shapes, and then apply these ideas to other pattern pairs in Sections 5 and 6. Section 7 contains a bijection that strengthens two of the equivalences.

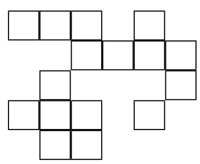
In total, we describe the relations between 13 pattern pairs. Of the resulting 156 equivalences, 10 are believed to be described fully — when a shape does not fit the assumptions of the theorem, the equivalence is expected to fail. This claim is based on empirical evidence; making it rigorous is expected to be difficult. We conjecture 4 more equivalences between 8 patterns; empirically, these hold for a broad class of shapes. Computations also suggest that the equivalences in this paper, proved and conjectured, are the "major" ones. We have not found any others that occur for large, fairly irregular classes of shapes. A proof of the conjecture, and several refinements of the equivalences in this paper would therefore complete the classification of the major equivalences.

In Section 8 especially, and also throughout the paper, we suggest a number of other interesting problems in this framework. Certainly, many more questions are waiting to be asked.

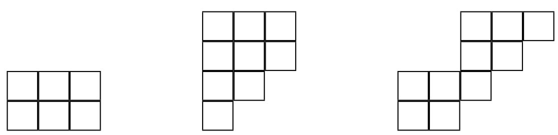
2. Basic Definitions

2.1. Shapes and Fillings.

Definition 2.1. An $m \times n$*-grid shape* S is a subset of boxes selected from a $m \times n$ 2-dimensional square grid. Here is a 5×6 grid shape S:

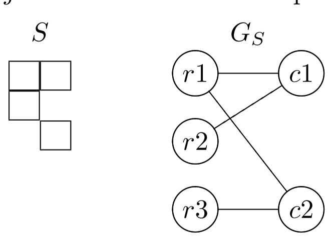


Some nice special cases are: rectangles, Young diagrams, and skew-shapes:

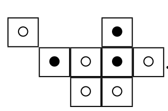


From now on, we will call these simply *shapes*.

Definition 2.2. The graph G_S of an $m \times n$-shape S is a bipartite graph on $m + n$ vertices. The first part of G_S, with m vertices, corresponds to the rows of S. The second part, with n vertices, corresponds to the columns. There is an edge in G_S between row i and column j iff S has a box in that position.

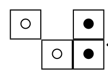


Definition 2.3. A *filling* F of a shape S places $\circ$ or $\bullet$ (alternative notations: $\circ$ or $\times$, 0 or 1) in every cell of the shape:



Notation 2.4. Let c be a cell of a shape S or filling F. Then, $S\backslash c$ and $F\backslash c$ denote the shape or filling with the cell removed.

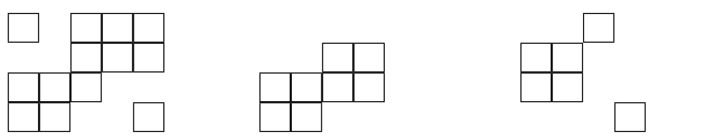
Definition 2.5. Deleting some rows and columns from a shape or filling makes a *minor*. Given a shape S, take subsets $R = \{r_1 < r_2 < \cdots < r_k\}$ of its rows, and $C = \{c_1 < c_2 < \cdots < c_l\}$ of its columns. The *minor* $M_{R,C}(S)$ of a shape S is a $k \times l$ shape which has a box in position i, j iff S has a box in position r_i, c_j. A minor of a filling also copies the contents of those boxes. Here is a $\{1, 2\} \times \{1, 3, 4\}$ minor of the filling above:



The minors of S are exactly the induced subgraphs of G_S.

Definition 2.6. A $k \times k$-*step* in a graph consists of moving from a cell c to another cell d so that both are in the same $k \times k$ minor with all k^2 cells present. A shape is $k \times k$-*connected* if one can get from any cell to any other cell by a sequence of $k \times k$-steps. A shape need not be topologically connected to be $k \times k$-connected:

2×2-connected $\quad$ 1×1-connected $\quad$ not $k \times k$-connected



For the purpose of 2×2 pattern avoidance, it will be enough to prove theorems for 2×2-connected shapes. Such shapes have a 4-cycle-connected G_S — one can get from any edge to any other edge by walking along adjacent edges of 4-cycles.

Definition 2.7. A grid shape S is *horizontally connected* if, after removing the shape's empty columns, the cells in every row form a single, unbroken block. *Vertical connectivity* is analogous. A shape is *connected* if it is both horizontally and vertically connected. The examples from Definition 2.6 are: (i) 2×2-connected, but horizontally- and vertically-disconnected; (ii) 1×1-connected, and connected (iii) $k \times k$-disconnected, but connected. So, neither condition implies the other.

2.2. **Patterns.** With permutations, there are several ways of defining pattern matching (avoidance). The standard definition says that a permutation contains a 132-pattern if its word $a_1 a_2 \dots a_n$ has the property that for some $i < j < k$, $a_j > a_k > a_i$. Other reasonable definitions happen to have a less interesting structure:

(1) For some $i < j < k$, the permutation's word has $a_i a_j a_k = 132$. Far fewer permutations match this pattern, and their number is equally easy to compute for all patterns.
(2) For some i, the permutation's word has $a_i a_{i+1} a_{i+2} = 132$. These are also easy to count, and are very few in number.

Our definition of patterns for fillings generalizes standard permutation pattern-avoidance. A permutation $\sigma \in S_n$ can be represented as a "rook diagram". We take an $n \times n$ shape and fill it with $\circ$s, except for a $\bullet$ in each cell i, j such that

$\sigma i = j$. So, 43152 becomes:

$$\begin{array}{|c|c|c|c|c|}\hline \circ & \circ & \bullet & \circ & \circ \\ \hline \circ & \circ & \circ & \circ & \bullet \\ \hline \circ & \bullet & \circ & \circ & \circ \\ \hline \bullet & \circ & \circ & \circ & \circ \\ \hline \circ & \circ & \circ & \bullet & \circ \\ \hline \end{array}$$

This permutation contains 132, namely 152 in positions 345. Correspondingly, the $\{1, 2, 5\} \times \{3, 4, 5\}$ minor of the rook diagram is equal to

$$\begin{array}{|c|c|c|}\hline \bullet & \circ & \circ \\ \hline \circ & \circ & \bullet \\ \hline \circ & \bullet & \circ \\ \hline \end{array}$$

A permutation is 132-avoiding if and only if its rook diagram avoids this minor. The generalization of such patterns to arbitrary shapes and fillings is what you would expect:

Definition 2.8. A *pattern* p is simply a filling of a shape. A filling F contains the pattern p if some minor of F equals p.

There are, just like for permutations, alternative definitions. Some are mentioned in Subsection 1.2. Here is another that we have not seen in the literature. A filling F contains the pattern p if some minor of F with continuous row and column sets is equal to p. Just like for permutations, there are fewer fillings containing a given pattern, and computations are a little easier because one does not have to check $O(m^2 n^2)$ minors. Pattern-avoidance is a local constraint in this definition, and should be more closely related to the geometry of the shape. This alternative definition might be worth investigating in detail, but it is beyond the scope of this paper.

Returning to Definition 2.8, the natural question is to characterize the number of fillings of a fixed shape S that avoid (or contain) certain p. We will focus on a specialization of this problem.

Definition 2.9. A 2×2-*pattern pair*, further called *pattern pair,* or *pp* for short, is an unordered pair of 2×2 patterns. For example: $\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)$. A filling F avoids a pattern pair $(p_1|p_2)$ if it contains neither p_1 nor p_2. We will call F a P*-avoiding filling*, or a P*-paf* for short.

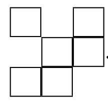
Let's fix a shape S and a pattern pair P. In this paper, we will not count the number of fillings of S that avoid P (although that is an interesting question in its own right). Instead, we will describe a number of equivalent pps:

Definition 2.10. $P_1 = (p_1^1|p_2^1)$ and $P_2 = (p_1^2|p_2^2)$ are *equivalent* if the fillings that avoid P_1 are equinumerous with P_2-pafs. Equally well, one may require the fillings *containing* these pps to be equinumerous.

Now it is clear why we may prove theorems only for 2×2-connected shapes:

Remark 2.11. A 2×2-disconnected shape contains a pattern iff one of its components does.

A special case is when the component has exactly one cell. We call such cells *detached.* This shape consists entirely of detached cells:



Each detached cell can be filled independently of all other cells, and therefore simply doubles the number of fillings avoiding the given pattern pair.

The first observation about equivalent pps is that we can take a "complement" of any pp, by replacing $\circ$ with $\bullet$, and vice-versa. For example, the complement of $P = \left(\begin{smallmatrix}\circ & \bullet\\ \circ & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \bullet\\ \bullet & \circ\end{smallmatrix}\right)$ is $\bar{P} = \left(\begin{smallmatrix}\bullet & \circ\\ \bullet & \bullet\end{smallmatrix}\middle|\begin{smallmatrix}\circ & \circ\\ \circ & \bullet\end{smallmatrix}\right)$.

Fact 2.12. *Given a shape S and a pp P, P-pafs are equinumerous with $\bar{P}$-pafs. The bijection is obvious: F is a P-paf iff $\bar{F}$ is a $\bar{P}$-paf.*

Since P and $\bar{P}$ are necessarily equivalent, we will identify every pp with its complement. Now, in order to enumerate all pps, we will number the single patterns. The pattern

$$p = \begin{matrix} a & b \\ c & d \end{matrix}$$

will be assigned number $n(p) = a + 2b + 4c + 8d = dcba_2$, where $\circ$ is 0 and $\bullet$ is 1. Then, $n(\bar{p}) = 15 - p = 1111_2 - dcba_2$ is the number of its complement. For consistency, we will write pps as $(p_1|p_2)$ with $n(p_1) < n(p_2)$.

There are $2^4 = 16$ single patterns, and consequently $\frac{16\cdot 15}{2} = 120$ pattern pairs. There are 8 self-complementary pattern pairs (this happens whenever $n(a)+n(b) = 15$). So, after identifying complements, we are left with $\frac{120-8}{2} + 8 = 64$ classes of pattern pairs.

3. Pattern Pair Equivalence

We will attempt to answer two related questions:

(1) For a given class of shapes, identify the sets of pps that are always equivalent for those shapes.
(2) Given two pps, identify the class of shapes, for which they are equivalent.

Obviously, having a complete answer to the second problem would solve the first as well. Restricting one's attention to a specialized class like Young diagrams might make proving equivalences easier. However, we have not found this to be the case. Therefore, for each pair of pps, this paper describes a class of shapes, for which the pps are equivalent. Outside of this class, we are not able to say that two patterns will *not* be equivalent for a given shape. Such a result may not be possible — it seems very hard to rule out numerical coincidences, which would lead to sporadic exceptions. For each of our equivalences, we will say whether, empirically, there are many cases of equinumerous fillings of shapes outside the described class. Table 1 on page 8 summarizes the pp equivalences characterized in this paper.

There are four distinct equivalence proofs in the paper, corresponding to the four parts of the table. We get the 14 results simply by rotating the pattern pairs. This is possible because $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ\\ \circ & \bullet\end{smallmatrix}\right)$ is a very symmetrical pp. It is invariant under row swaps, column swaps, and transposition. In fact, $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ\\ \circ & \bullet\end{smallmatrix}\right)$-avoidance is a property of the shape's graph, oriented according to the filling. Consequently, the argument for counting its pafs is invariant under all these symmetries. On the other hand, the shapes on the left-hand sides of the table's equivalences are altered by the above transformations, and we get 4 variants of each, with completely symmetrical proofs.

Section 5			Section 4			Section 6		
$\left(\begin{array}{cc\|cc} \circ & \circ & \bullet & \circ \\ \circ & \circ & \circ & \circ \end{array}\right)$ (0|1)	$\leftrightarrow$	$\left(\begin{array}{cc\|cc} \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \circ & \bullet \end{array}\right)$ (6|9)	$\left(\begin{array}{cc\|cc} \circ & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \bullet \end{array}\right)$ (8|9)	$\leftrightarrow$	$\left(\begin{array}{cc\|cc} \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \circ & \bullet \end{array}\right)$ (6|9)	$\left(\begin{array}{cc\|cc} \bullet & \circ & \circ & \circ \\ \circ & \circ & \bullet & \circ \end{array}\right)$ (1|4)	$\leftrightarrow$	$\left(\begin{array}{cc\|cc} \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \circ & \bullet \end{array}\right)$ (6|9)
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$\left(\begin{array}{cc\|cc} \circ & \circ & \circ & \circ \\ \circ & \circ & \bullet & \circ \end{array}\right)$ (0|4)	$\leftrightarrow$	$\left(\begin{array}{cc\|cc} \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \circ & \bullet \end{array}\right)$ (6|9)	$\left(\begin{array}{cc\|cc} \circ & \bullet & \circ & \bullet \\ \circ & \circ & \bullet & \circ \end{array}\right)$ (2|6)	$\leftrightarrow$	$\left(\begin{array}{cc\|cc} \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \circ & \bullet \end{array}\right)$ (6|9)	$\left(\begin{array}{cc\|cc} \circ & \circ & \circ & \circ \\ \bullet & \circ & \circ & \bullet \end{array}\right)$ (4|8)	$\leftrightarrow$	$\left(\begin{array}{cc\|cc} \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \circ & \bullet \end{array}\right)$ (6|9)
$\left(\begin{array}{cc\|cc} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet \end{array}\right)$ (0|8)	$\leftrightarrow$	$\left(\begin{array}{cc\|cc} \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \circ & \bullet \end{array}\right)$ (6|9)	$\left(\begin{array}{cc\|cc} \bullet & \circ & \bullet & \circ \\ \circ & \circ & \circ & \bullet \end{array}\right)$ (1|9)	$\leftrightarrow$	$\left(\begin{array}{cc\|cc} \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \circ & \bullet \end{array}\right)$ (6|9)	$\left(\begin{array}{cc\|cc} \circ & \bullet & \circ & \circ \\ \circ & \circ & \circ & \bullet \end{array}\right)$ (2|8)	$\leftrightarrow$	$\left(\begin{array}{cc\|cc} \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \circ & \bullet \end{array}\right)$ (6|9)

Section 7		
$\left(\begin{array}{cc\|cc} \bullet & \circ & \bullet & \circ \\ \circ & \circ & \circ & \bullet \end{array}\right)$ (1|9)	$\leftrightarrow$	$\left(\begin{array}{cc\|cc} \circ & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \bullet \end{array}\right)$ (8|9)
$\left(\begin{array}{cc\|cc} \circ & \bullet & \circ & \bullet \\ \circ & \circ & \bullet & \circ \end{array}\right)$ (2|6)	$\leftrightarrow$	$\left(\begin{array}{cc\|cc} \circ & \circ & \circ & \bullet \\ \bullet & \circ & \bullet & \circ \end{array}\right)$ (4|6)

TABLE 1. Pattern pair equivalences described in this paper, by section. In order to make the symmetry clearer, we use $\left(\begin{array}{cc|cc} \circ & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \bullet \end{array}\right)$ instead of $\left(\begin{array}{cc|cc} \circ & \bullet & \bullet & \bullet \\ \bullet & \circ & \bullet & \circ \end{array}\right)$ mentioned in the introduction.

The top three parts of the table are arranged by the strength of the required assumptions, strongest to weakest from left to right. See the corresponding sections and Subsection 8.3 for the details.

Although the equivalences in Section 7 follow by combining two equivalences from Section 4, the proof in Section 7 works for more general shapes, and is bijective. Empirically, too, the patterns from Section 7 can be equivalent in shapes where neither is equivalent to $\left(\begin{array}{cc|cc} \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \circ & \bullet \end{array}\right)$.

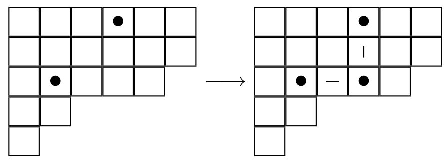
Regarding empirical data — we have computed pp equivalence classes for a substantial number of shapes of varying degrees of "regularity". It is difficult to make sharp conjectures about classes of shapes based on these data, because a class of shapes can have a rather opaque definition (see e.g. Theorem 4.10 and Definition 4.1). However, we present some observations and guesses in Section 8.

3.1. **A note about pattern and shape symmetry.** Before describing the equivalences in Table 1 on page 8, we should point out one source of trivial equivalences. There are two structure-preserving transformations on both patterns and shapes: row order reversal, and transposition (column reversals are row reversals conjugated by transposition). If one pp is mapped to another pp by some combination T of these transformations, then these pps will be equivalent in all shapes that have a T-symmetry. The possible symmetries T are: the 90°-, 180°-, 270°-degree rotations, and reflections across the horizontal, vertical, and two diagonal axes. Most shapes are not symmetric; these equivalences, although easy to list, usually do not apply.

4. $\left(\begin{array}{cc|cc} \circ & \bullet & \bullet & \bullet \\ \bullet & \circ & \bullet & \circ \end{array}\right)$ AND $\left(\begin{array}{cc|cc} \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \circ & \bullet \end{array}\right)$: ⅃-DIAGRAMS AND ACYCLIC ORIENTATIONS

This pair of pps started it all. Their equivalence was first proved by Alex Postnikov [7], using a recurrence for ⅃-diagrams found by Lauren Williams [9] and his analogous recurrence for acyclic orientations.

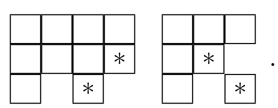
4.1. **⅃-diagrams.** Originally, a ⅃-diagram was defined to be a binary filling of a Young diagram having the ⅃-property: if two cells located at the bottom-left and top-right corners of a rectangle contain $\bullet$, then the cell at the bottom-right corner (making a "⅃" shape) must also contain $\bullet$:



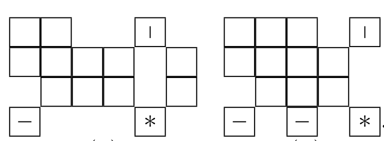
In our terminology, a ⅃-diagram is a $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \bullet\\ \bullet & \circ\end{smallmatrix}\right)$-avoiding filling. This definition is valid for all shapes. There is a caveat here: in a Young diagram, one always has the upper-left cell of a 2×2 minor. In general, it is quite possible that it's missing, but our definition requires this cell to be present. Another way to interpret the ⅃-property is that the upper-left cell is irrelevant, and need not even be present. We chose to have a complete 2×2 minor because this naturally preserves the connection between acyclic orientations and ⅃-diagrams — the proof for acyclic orientations *requires* the upper-left cell to be present. Nonetheless, it would be an interesting generalization to permit incomplete minors; [3, 2] have some related results.

Lauren Williams introduced the polynomial $F_S(q)$, where the coefficient of q^k counts the number of $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \bullet\\ \bullet & \circ\end{smallmatrix}\right)$-avoiding fillings of shape S that contain k $\bullet$s. She gave a simple recurrence for $F_S(q)$ in Young diagrams. We will now see that this recurrence generalizes to a much larger class of shapes.

The recurrence starts at a bottom-right corner c of the shape — that is, the cell must be rightmost in its row, and bottommost in its column. In the two shapes below, the cells marked with $*$ are such corners:



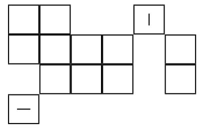
The cells above c and the cells to the left of c, ignoring discontinuities, form the bottom and right edges of a rectangle. In these shapes, the corner c is marked with $*$, while the edges are marked with dashes:



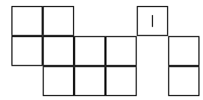
(a) (b)

In order for the recurrence to work, all the cells in this rectangle must be present as in (a) and (b). We will call such rectangles *complete*. Because of Remark 2.11, we will also require the shape to be 2×2-connected. To compute $F_S(q)$, the recurrence requires the values of $F_S(q)$ on four smaller shapes, illustrated on the example (a) above:

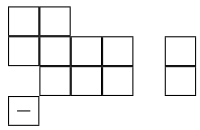
S_1 – delete corner



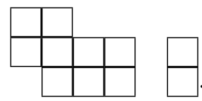
S_2 – delete c's row



S_3 – delete c's column



S_4 – delete both

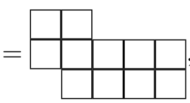


These deletions may render the shape 2×2-disconnected. To compute F_{S_i}, we will split S_i into its 2×2-connected components $S_i^{(j)}$, each a separate shape, and multiply their polynomials:

$$F_{S_i}(q) = \prod_j F_{S_i^{(j)}}(q). \tag{2}$$

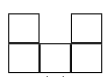
Some of the components will be detached cells, each of which will contribute a factor of $(1 + q)$, because it may be filled with either $\circ$ or $\bullet$, independently of any other cell. In our example, the four shapes simplify to one (modulo detached cells, which are responsible for the $(1 + q)^i$ factors):

$$S_1^{(1)} = S_2^{(1)} = S_3^{(1)} = S_4^{(1)} =$$

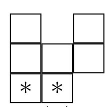


$$\begin{array}{cc} F_{S_1} = (1+q)^2 F_{S_1^{(1)}} & F_{S_2} = (1+q) F_{S_2^{(1)}} \\ F_{S_3} = (1+q) F_{S_3^{(1)}} & F_{S_4} = F_{S_4^{(1)}} \end{array}.$$

Now, the recurrence (which we will define in Lemma 4.3) may be used to compute F for every $S_i^{(j)}$. The decomposition of S_i into $S_i^{(j)}$ is necessary to cover a larger class of shapes. Without taking 2×2-components, it would be impossible to apply the recurrence to shapes like (c), and to compute F for shapes like (d) that reduce to (c):



(c)



(d)

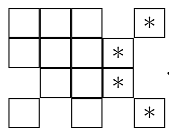
This points out an important problem: for some shapes, we may be unable to repeatedly apply the recurrence all the way down to the empty shape. The worst case is repeated expansion along S_1 (delete corner); in order for it to succeed, this definition must apply:

Definition 4.1. A shape is *bottom-right complete rectangle-erasable* (CR-erasable for short) if all of its 2×2-components satisfy the following recursive rule. In each component $S^{(j)}$, considered as a separate shape, we can find a special bottom-right corner c with two properties. Firstly, the corner must have a complete rectangle. Secondly, the shape $S^{(j)} \backslash c$ must be bottom-right CR-erasable. As the base case, the empty shape is CR-erasable.

Lemma 4.2. *If $S^{(j)}$ is a bottom-right CR-erasable 2×2-connected component, use the special corner c to obtain S_1, S_2, S_3, and S_4. Then, the S_i are bottom-right CR-erasable.*

Proof. S_1 is CR-erasable by definition. In S_2, we have deleted a whole row R; nonetheless, it is still CR-erasable. To see this, erase S_1 and S_2 in lockstep. Let d be the cell about to be deleted in S_1 (actually, a sub-sub-...-sub-component of S_1, because the deletion process fragments the shape). If d was in R, then there is nothing to do in S_2. Otherwise, d still has a complete rectangle in the sub-...-component of S_2, because deleting a whole row cannot make a rectangle incomplete. The argument showing that S_3 and S_4 are CR-erasable is analogous. $\square$

The shape (b) introduced above is not bottom-right CR-erasable. We can eliminate the following cells marked with $*$, and will get stuck with a single corner that has an incomplete rectangle:



.

The recurrence cannot be computed, and indeed, for this shape there are 5566 $\left(\begin{smallmatrix} \circ & \bullet \\ \bullet & \circ \end{smallmatrix} \middle| \begin{smallmatrix} \bullet & \bullet \\ \bullet & \circ \end{smallmatrix}\right)$-pafs, but 5476 $\left(\begin{smallmatrix} \circ & \bullet \\ \bullet & \circ \end{smallmatrix} \middle| \begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix}\right)$-pafs. The reader should check that, in contrast, the recurrence succeeds for the shape (a).

We have now completely described what the recurrence needs to compute $F_S(q)$. It remains to describe how it works:

Lemma 4.3. *If the shape S is bottom-right CR-erasable, then the generating polynomial of $\left(\begin{smallmatrix} \circ & \bullet \\ \bullet & \circ \end{smallmatrix} \middle| \begin{smallmatrix} \bullet & \bullet \\ \bullet & \circ \end{smallmatrix}\right)$-avoiding fillings of S with k $\bullet$s can be computed using* only *the recurrence*

$$F_S(q) = \prod_j \left(qF_{S_1^{(j)}}(q) + F_{S_2^{(j)}}(q) + F_{S_3^{(j)}}(q) - F_{S_4^{(j)}}(q) \right), \tag{3}$$

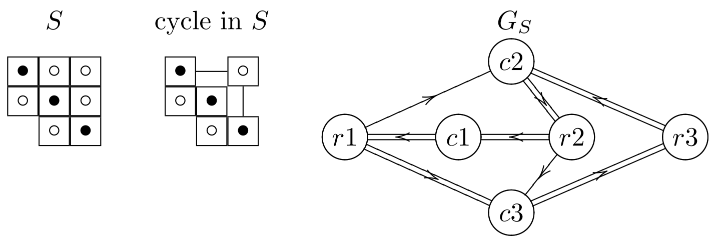
where $S^{(j)}$ are the 2×2-connected components of S considered as separate shapes, and $S_i^{(j)}$ are copies of $S^{(j)}$ after deleting the special corner, the cells in its row, column, and row plus column, as discussed above. The initial condition is $F_\emptyset(q) = 1$.

Proof. Most of the proof is done already: we saw that F_S is a product over 2×2-connected components, and we showed that the $S_i^{(j)}$ are CR-erasable. It remains to explain the expression inside the parentheses.

Consider a particular $S^{(j)}$; by Definition 4.1, it has a special bottom-right corner c with a complete rectangle (the next cell to be deleted). If the corner contains $\bullet$, then no forbidden pattern can involve this corner (because both patterns have $\circ$ in the bottom-right corner). So, the number of such pafs is $F_{S_1^{(j)}}(q)$, and we multiply it by q to account for $\bullet$ in the corner.

If the corner contains $\circ$, this constrains the cells above and to the left of c. Since the shape contains c's complete rectangle, $\bullet$ must not be simultaneously present in both the column above and in the row to the left of c. Either c's row or c's column must consist entirely of $\circ$s. If it is the row, then it cannot participate in a forbidden pattern — both patterns have at least one $\bullet$ in each row. The number of ways to fill the remaining cells is enumerated by $F_{S_2^{(j)}}(q)$. The reasoning for the column case is identical, and that contributes $F_{S_3^{(j)}}(q)$. However, this double-counts the case when both the row and the column are filled with $\circ$s, so we subtract $F_{S_4^{(j)}}(q)$. $\square$

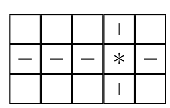
4.2. **Acyclic orientations.** Recall from Definition 2.2 that a bipartite graph G_S corresponds to each shape S. A filling of the shape gives an orientation: the edge points from a row to a column if its cell contains $\circ$, and from a column to a row otherwise. A cycle in this graph corresponds to a sequence of cells in the filling alternating between "same column, different row" and "same row, different column", with contents (independently) alternating between $\circ$ and $\bullet$. Here is an example:

(4)


One of the graph's cycles is shown with double lines, and the cells of S that correspond to edges of the cycle are isolated in the middle shape.

Note that in this example, there is also a 4-cycle: $c1 - r1 - c2 - r2$, which corresponds to the minor $\{r1, r2\} \times \{c1, c2\}$ in the filling. In fact, every 4-cycle is a 2×2 minor filled with one of the two patterns $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ\\ \circ & \bullet\end{smallmatrix}\right)$. So, 4-acyclic fillings are exactly the $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ\\ \circ & \bullet\end{smallmatrix}\right)$-avoiding fillings. We will soon see that 4-cycles are present in every cyclic filling of a large class of shapes. But first, we need to extend the "complete rectangle" that we defined for bottom-right corners of ⅃-diagrams:

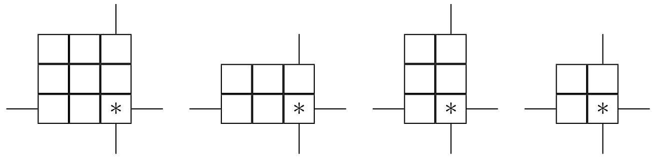
Definition 4.4. A cell c in shape S has a *complete rectangle (CR)* if for every choice of c_r from c's row, and c_c from c's column, there is a cell c_{rc} in S at the intersection of c_c's row and c_r's column. Here is an example, with c marked by $*$, the c_rs marked by $-$, c_cs marked by $|$, and the c_{rc}s left blank:



This implies that each choice of c, c_r, c_c, c_{rc} is a 2×2 minor, and that the vertices of the edges adjacent to c in G_S induce a complete bipartite subgraph. We will call such cells *CR-cells.*

Lemma 4.5. *Let S be a 2×2-connected shape with a CR-cell c. Then $S' = S \backslash c$ has at most three 2×2-connected components, all but one of which are detached cells that are leaf edges of $G_{S'}$.*

Proof. Let n_r and n_c be the number of cells (including c) in c's row and column, respectively. Both are at least 2 — otherwise c wouldn't be 2×2-connected. There are 4 cases: $n_r, n_c > 2$; $n_r = 2, n_c > 2$; $n_r > 2, n_c = 2$; $n_r = n_c = 2$. We will use some pictures to illustrate them, and will always place c as the bottom-right-most cell. This is legitimate, because 2×2-connectivity is a property of G_S, and as such is invariant under rearrangements of rows and columns. Here are rectangles of c in each case (> 3 neighbors in either direction works just like 3):



The "$*$" marks c, while "$-$" and "$|$" emphasize that there are no other cells in those rows and columns. In the first case, removing c will leave a 2×2-connected shape. If that were not true, some two cells d and e that are in a 2×2 minor together with c would become disconnected. But, every 2×2 minor involving c belongs to its rectangle. With $n_r > 2, n_c > 2$, the rectangle remains 2×2-connected after deleting c, and no such d and e exist.

The second and third case are symmetric, so we will cover only $n_r = 2, n_c > 2$. The cell c_c in c's column becomes detached — after c is deleted, c_c is the only cell in its column, and thus cannot be in a 2×2 minor (one component). Moreover, c_c's column is a leaf vertex in $G_{S'}$, because c_c is its only edge. The other cells in the rectangle stay interconnected through minors not involving c. So, the rest of the filling, $S' \backslash c_c$, is 2×2-connected (a second component).

The fourth case is not much different from the second and third. The cells c_r and c_c are left alone in their row and column, respectively, and therefore become detached (two components, two leaf edges). The connections of the remaining cell c_{rc} to the rest of the shape are intact, and $S' \backslash \{c_r, c_c\}$ is therefore the third component.

□

Now, we extend the notion CR-erasability from Definition 4.1 to allow cells other than bottom-right corners.

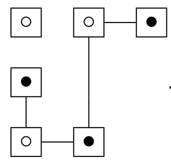
Definition 4.6. Let P be a cell predicate, such as "bottom-right"; we will omit P to mean "any". A shape is P *complete rectangle-erasable* (P CR-erasable for short) if all of its 2×2-components satisfy the following recursive rule. In each component $S^{(i)}$, considered as a separate shape, we can find a special CR-cell c satisfying P, such that $S^{(i)} \backslash c$ is P CR-erasable. The empty shape is P CR-erasable.

From Lemma 4.5, we see that if a 2×2-connected shape S is CR-erasable, the deletion procedure is particularly simple. First, we delete some CR-cell from S. Then, we delete the resulting detached cells, and we are once again left with a 2×2-connected CR-erasable shape.

Lemma 4.7. *Suppose that S is a 2×2-connected, CR-erasable shape. Then, any cyclic filling of S contains a 4-cycle.*

Proof. Let F be a filling of S containing a cycle C. Let c be the special CR-cell given by by CR-erasability.

Case 1: Suppose that c belongs to the cycle C. Then, the row and column of c must each contain another cell from the cycle — c_r and c_c, respectively. Without loss of generality, we may assume that c is filled with $\bullet$ (otherwise, take the complementary filling – it will have the same cycles). Thus, c_r and c_c are filled with $\circ$. Now, consider the cell c_{rc} in c_r's column, and c_c's row (it exists because c has a complete rectangle). If c_{rc} is filled with $\bullet$, the four cells c, c_r, c_c, c_{rc} are a 4-cycle, and we are done. So, assume that c_{rc} is filled with $\circ$. Since cycles alternate rows and columns, there must be a further element of the cycle c'_r in c_r's column. Similarly, we get c'_c in c_c's row. Both c'_r and c'_c must be filled with $\bullet$. In this illustration, the bottom-right corner is c, the detached cell is c_{rc}:


.

Therefore, we can replace $c'_r - c_r - c - c_c - c'_c$ by $c'_r - c_{rc} - c'_c$ in C to obtain a shorter cycle C', which avoids c. This brings us to case 2.

Case 2: If c does not belong to the cycle, we can delete it. This might create a detached cell in c's row or column, or a cell in each, as in the proof of Lemma 4.5. Suppose c_c is the detached cell from c's column. Then, c_c could not have been in C either, because $c_c \in C$ implies that there is a second cell $d \in C$ in c_c's column. But, the only possibility for d is c, and $c \notin C$. An analogous argument shows that the cycle does not pass through the detached cell in c's row. Thus, the cycle lies entirely in the remaining 2×2-connected component F'. This new filling is strictly smaller, and satisfies our initial assumptions, so we may repeat the argument. After finitely many iterations, the number of cells will become ≤ 4, but the only cyclic filling on ≤ 4 cells is the 4-cycle. □

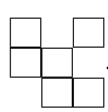
The proof of Lemma 4.7 can be modified slightly to obtain a result with different assumptions:

Corollary 4.8. *Suppose that the shape S can be erased by deleting a CR-cell, and repeating the procedure on the resulting shape (without breaking it into 2×2-components). The reader is invited to check that, in particular, this condition holds for connected shapes (see Definition 2.7). Then, a 4-acyclic filling of S is acyclic.*

Proof. We can use the unmodified "Case 1" from the above proof. If we end up making a shorter cycle that does not pass through c, we delete c. The resulting filling still has a deletion sequence of CR-cells, so we win by induction on the size of the filling. □

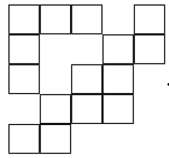
Although the hypotheses of Lemma 4.7 and Lemma 4.8 sound similar to Definition 4.1, they are not more or less general. Specifically, in these results, the cells that we delete do not have to be corners. But, this comes with extra assumptions: 2×2-connectedness, or erasability without decomposing into 2×2-components.

For shapes satisfying one of these conditions, acyclic orientations are the $\left(\begin{smallmatrix}\circ&\bullet\\ \bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\ \circ&\bullet\end{smallmatrix}\right)$-avoiding fillings. Like with ⅃-diagrams, this pattern-avoidance model is not perfect — in this case, if the shape is arbitrary, we cannot express acyclicity in terms of small minors. For example, one *needs* a 3×3 minor to detect a cycle in this shape (see the example (4)):

(5)  .

It has 64 $\left(\begin{smallmatrix}\circ&\bullet\\ \bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\ \circ&\bullet\end{smallmatrix}\right)$-avoiding (4-acyclic) fillings, and 62 acyclic ones. It is CR-erasable, but is not 2×2-connected, and cannot be CR-erased without splitting into 2×2-components. Thus, it demonstrates that the extra assumptions in Lemma 4.7 and

Lemma 4.8 are necessary. The following shape is 2×2-connected, but is not CR-erasable:



It also has more 4-acyclic fillings – 14894 – than acyclic ones – 13790. It might be an interesting combinatorial problem to describe the graphs of such shapes.

4.3. **Recurrence for $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ\\ \circ & \bullet\end{smallmatrix}\right)$-avoidance.** In Lemma 24.2 of [7], Alex Postnikov proved a recurrence for the chromatic polynomial $\chi_{G_\lambda}(t)$ of the graph of a Young diagram. He then specialized it to obtain a recurrence for the number of acyclic orientations of G_λ. We will generalize his result to all CR-erasable shapes. However, the chromatic polynomial does not decompose across 2×2-components. If it did, the recurrence for the chromatic polynomial would hold for all CR-erasable graphs, and we would get the same recurrence for $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ\\ \circ & \bullet\end{smallmatrix}\right)$-avoiding fillings as for acyclic fillings. But, this is impossible as shown by example (5).

Therefore, we have to specialize to $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ\\ \circ & \bullet\end{smallmatrix}\right)$-avoidance straight away.

Lemma 4.9. *Let A_S be the number of $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ\\ \circ & \bullet\end{smallmatrix}\right)$-pafs of shape S. If the shape S is CR-erasable, then A_S can be computed using* only *the recurrence*

$$A_S = \prod_j \left(A_{S_1^{(j)}} + A_{S_2^{(j)}} + A_{S_3^{(j)}} - A_{S_4^{(j)}} \right), \tag{6}$$

where $S^{(j)}$ are the 2×2-connected components of S considered as separate shapes, and $S_i^{(j)}$ are copies of $S^{(j)}$ after deleting the special cell, the cells in its row, column, and row plus column, just like in Lemma 4.3. The initial condition is $A_\emptyset = 1$.

Proof. We need to show that for every 2×2-connected shape $S^{(j)}$ with a CR-cell c, the number of $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ\\ \circ & \bullet\end{smallmatrix}\right)$-pafs is given by the quantity in the parentheses. The rest comes together just like in Lemma 4.3.

By Lemma 4.7, it is enough to compute the number of acyclic fillings of $S^{(j)}$. By Postnikov's Lemma 24.2 [7], the chromatic polynomial $\chi_{S^{(j)}}$ of $G_{S^{(j)}}$ can be written as

$$\chi_{S^{(j)}}(t) = \chi_{S_1^{(j)}}(t) - \frac{1}{t} \left(\chi_{S_2^{(j)}}(t) + \chi_{S_3^{(j)}}(t) - \chi_{S_4^{(j)}}(t) \right). \tag{7}$$

Technically, Postnikov's proof was written for a corner of a Young shape, not a CR-cell of a 2×2-connected shape. However, his proof uses only the structure of G_S, and disregards the positions of rows and columns. Therefore, it generalizes without modifications to any shape with a CR-cell. Further following Postnikov, we specialize (7) to $t = -1$, to obtain a relation in terms of the numbers of acyclic orientations $(-1)^n \chi_{S^{(j)}}(-1)$. The exponent n is the number of vertices in the graph of $S^{(j)}$, and because we only delete the edges, the graph of $S_i^{(j)}$. So, if ao_S is the number of acyclic fillings of shape S, we get

$$ao_{S^{(j)}} = ao_{S_1^{(j)}} + ao_{S_2^{(j)}} + ao_{S_3^{(j)}} - ao_{S_4^{(j)}}.$$

That is not quite the end — we need to show that acyclic fillings and $\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ\\ \circ & \bullet\end{smallmatrix}\right)$-pafs are equinumerous in the sub-shapes $S_i^{(j)}$ (for the left-hand side, we know this already). For $S_1^{(j)}$, look back at the proof of Lemma 4.5. The shape has one large CR-erasable 2×2-connected component, and at most two leaf edges. No cycle can pass through leaf edges, and the big component is okay by Lemma 4.7.

The proofs for $S_2^{(j)}$, $S_3^{(j)}$, and $S_4^{(j)}$ are slight modifications of the same argument, which we omit due to space limitations. Briefly, the shape remains 2×2-connected after these whole-row or whole-column deletions, and the proof that the new shape is CR-erasable is just like Lemma 4.2. Thus, Lemma 4.7 applies. □

4.4. **Equivalence of $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\bullet\\\bullet&\circ\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$, and its symmetries.** Following [7], we now specialize (3) with $q = 1$ to get a recurrence counting the number of $\lrcorner$-diagrams of shape S:

$$\lrcorner_S = \prod_j \left(\lrcorner_{S_1^{(j)}} + \lrcorner_{S_2^{(j)}} + \lrcorner_{S_3^{(j)}} - \lrcorner_{S_4^{(j)}} \right). \tag{8}$$

This holds for all bottom-right CR-erasable shapes. Such shapes are, of course, CR-erasable, and so the $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$ recurrence also applies. The recurrences are identical, and have the same initial conditions, $\lrcorner_\emptyset = A_\emptyset = 1$. To summarize:

Theorem 4.10. $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\bullet\\\bullet&\circ\end{smallmatrix}\right)$ *and its complement* $\left(\begin{smallmatrix}\circ&\circ\\\circ&\bullet\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$ *are equivalent to* $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$ *for every bottom-right CR-erasable shape.* □

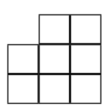
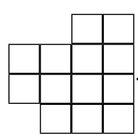
The recurrence for $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\bullet\\\bullet&\circ\end{smallmatrix}\right)$ depends on the bottom-right corner because the pattern pair has an asymmetry that makes this corner special. The recurrence for $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$ works in a far more general context, including any corner CR-erasability. If we rotate $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\bullet\\\bullet&\circ\end{smallmatrix}\right)$, a different corner type will become special. Then, we can rotate the proofs and definitions of Subsection 4.1 to obtain some symmetric results. The reader may wish to learn to read upside-down and sideways before continuing.

Theorem 4.11. *The analogs of Lemma 4.3,* (8)*, and Theorem 4.10 hold for:*

(1) $\left(\begin{smallmatrix}\circ&\circ\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\right)$ *and complement* $\left(\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\bullet\\\circ&\bullet\end{smallmatrix}\right)$ *in bottom-left CR-erasable shapes.*
(2) $\left(\begin{smallmatrix}\circ&\bullet\\\circ&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\right)$ *and complement* $\left(\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\bullet&\bullet\end{smallmatrix}\right)$ *in top-right CR-erasable shapes.*
(3) $\left(\begin{smallmatrix}\bullet&\circ\\\circ&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$ *and complement* $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ&\bullet\\\bullet&\bullet\end{smallmatrix}\right)$ *in top-left CR-erasable shapes.*

Other symmetries, like reflections, do not add any new pattern pairs to this list. □

Empirically, the characterization of these equivalences in terms of CR-erasable shapes does not look tight. There appears to be a sizable class of unexplained shapes that includes

 and .

These shapes have no CR bottom-right corners, but have equivalent $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\bullet\\\bullet&\circ\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$ nonetheless.

5. $\left(\begin{smallmatrix}\circ&\circ\\\circ&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\circ\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$: The Same Recurrence

We can get a recurrence for counting $\left(\begin{smallmatrix}\circ&\circ\\\circ&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\circ\end{smallmatrix}\right)$-pafs by exactly the same method. There is only a small difference in the way the counts are refined by the contents of the fillings:

Lemma 5.1. *If the shape S is bottom-right CR-erasable, then $O_S(q)$, the generating polynomial of $\left(\begin{smallmatrix} \circ & \circ \\ \circ & \circ \end{smallmatrix}\middle|\begin{smallmatrix} \bullet & \circ \\ \circ & \circ \end{smallmatrix}\right)$-avoiding fillings of S with k $\circ$s can be computed using* only *the recurrence*

$$O_S(q) = \prod_j \left(O_{S_1^{(j)}}(q) + q\left(O_{S_2^{(j)}}(q) + O_{S_3^{(j)}}(q) - O_{S_4^{(j)}}(q)\right)\right), \tag{9}$$

where $S^{(j)}$ are as in Lemma 4.3, and the initial condition is $O_\emptyset(q) = 1$.

Proof. The proof is completely analogous to that of Lemma 4.3. If the corner contains $\bullet$, we are free to delete it — this makes the $S_1^{(j)}$ term. Otherwise, the corner's row or column (or both) consists entirely of $\bullet$s. Such a row or column cannot participate in either pattern of the pp. So, we get the remaining three terms (the deleted $\circ$ corner gives the factor of q). □

We specialize the recurrence with $q = 1$, rotate the pattern pair, and get all the analogous equivalences:

Theorem 5.2. *Not listing complements, the following pps are equivalent to $\left(\begin{smallmatrix} \circ & \bullet \\ \bullet & \circ \end{smallmatrix}\middle|\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix}\right)$: for bottom-right corner CR-erasable shapes — $\left(\begin{smallmatrix} \circ & \circ \\ \circ & \circ \end{smallmatrix}\middle|\begin{smallmatrix} \bullet & \circ \\ \circ & \circ \end{smallmatrix}\right)$, bottom-left — $\left(\begin{smallmatrix} \circ & \circ \\ \circ & \circ \end{smallmatrix}\middle|\begin{smallmatrix} \circ & \bullet \\ \circ & \circ \end{smallmatrix}\right)$, top-left — $\left(\begin{smallmatrix} \circ & \circ \\ \circ & \circ \end{smallmatrix}\middle|\begin{smallmatrix} \circ & \circ \\ \circ & \bullet \end{smallmatrix}\right)$, top-right — $\left(\begin{smallmatrix} \circ & \circ \\ \circ & \circ \end{smallmatrix}\middle|\begin{smallmatrix} \circ & \circ \\ \bullet & \circ \end{smallmatrix}\right)$.* □

Empirically, we found no shape with equivalent $\left(\begin{smallmatrix} \circ & \circ \\ \circ & \circ \end{smallmatrix}\middle|\begin{smallmatrix} \bullet & \circ \\ \circ & \circ \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \circ & \bullet \\ \bullet & \circ \end{smallmatrix}\middle|\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix}\right)$, such that the shape is not bottom-right CR-erasable.

6. $\left(\begin{smallmatrix} \bullet & \circ \\ \circ & \circ \end{smallmatrix}\middle|\begin{smallmatrix} \circ & \circ \\ \bullet & \circ \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \circ & \bullet \\ \bullet & \circ \end{smallmatrix}\middle|\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix}\right)$: The Same Recurrence With a Twist

The pair $\left(\begin{smallmatrix} \bullet & \circ \\ \circ & \circ \end{smallmatrix}\middle|\begin{smallmatrix} \circ & \circ \\ \bullet & \circ \end{smallmatrix}\right)$ requires two changes. Firstly, if this pp is present in a shape S, then it also belongs to any S' obtained by permuting rows of S. This is because swapping the rows of the pp does not change it. So, the relevant requirement for a cell c this time a *right complete rectangle* — c is a CR-cell that is rightmost in its row. The recurrence lemma in this case does not give a nice way to count pafs by the number of $\circ$s or $\bullet$s they contain.

Lemma 6.1. *If the shape S is right CR-erasable, then the number R_S of $\left(\begin{smallmatrix} \bullet & \circ \\ \circ & \circ \end{smallmatrix}\middle|\begin{smallmatrix} \circ & \circ \\ \bullet & \circ \end{smallmatrix}\right)$-pafs of S can be computed using* only *the recurrence*

$$R_S = \prod_j \left(R_{S_1^{(j)}} + R_{S_2^{(j)}} + R_{S_3^{(j)}} - R_{S_4^{(j)}}\right), \tag{10}$$

where $S^{(j)}$ are as in Lemma 4.3, and the initial condition is $R_\emptyset = 1$.

Proof. Again, we need to justify the decomposition into the four sub-shapes. If c is the special right CR-cell, and it contains $\bullet$, it is not involved in forbidden patterns and can be deleted to make $S_1^{(j)}$. Now comes the second change from the previous proofs. If c contains $\circ$, and another cell d in its column contains $\circ$, then every other cell in c's row must be identical to the corresponding cell in d's row. Indeed if the two cells in one column were different, we would get a pattern from $\left(\begin{smallmatrix} \bullet & \circ \\ \circ & \circ \end{smallmatrix}\middle|\begin{smallmatrix} \circ & \circ \\ \bullet & \circ \end{smallmatrix}\right)$. Thus, there are two cases: either c's column consists of $\bullet$s, or c's row is fully replicated by another row. In the first case, no cells in the column can participate in forbidden patterns, and we can delete the column to make $S_3^{(j)}$. In the second case, if a forbidden pattern involves c's row, there is a copy of this pattern using d's row instead of c's row. So, the pattern would have to be present in $S_2^{(j)}$ to show up in $S^{(j)}$. Therefore, we can delete the bottom row, and count fillings of $S_2^{(j)}$. Just as before, this double-counts the case where the bottom row is replicated by some other row *and* the column consists of $\bullet$s; that's $S_4^{(j)}$. □

Theorem 6.2. *Omitting complements, the following pps are equivalent to* $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$*: for right CR-erasable shapes —* $\left(\begin{smallmatrix}\bullet&\circ\\\circ&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ&\circ\\\bullet&\circ\end{smallmatrix}\right)$*, left —* $\left(\begin{smallmatrix}\circ&\bullet\\\circ&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ&\circ\\\circ&\bullet\end{smallmatrix}\right)$*, top —* $\left(\begin{smallmatrix}\circ&\circ\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ&\circ\\\circ&\bullet\end{smallmatrix}\right)$*, bottom —* $\left(\begin{smallmatrix}\bullet&\circ\\\circ&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ&\bullet\\\circ&\circ\end{smallmatrix}\right)$*.* □

Empirically, we found no shape such that $\left(\begin{smallmatrix}\circ&\bullet\\\circ&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ&\circ\\\circ&\bullet\end{smallmatrix}\right)$ is equivalent to $\left(\begin{smallmatrix}\circ&\bullet\\\bullet&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$, but the shape is not left CR-erasable.

7. $\left(\begin{smallmatrix}\bullet&\circ\\\circ&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\circ&\circ\\\circ&\bullet\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$: A Bijection

If a shape's 2×2-connected components are horizontally- and vertically-connected (Definition 2.7), the following bijection identifies $\left(\begin{smallmatrix}\bullet&\circ\\\circ&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$-avoiding fillings with $\left(\begin{smallmatrix}\circ&\circ\\\circ&\bullet\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$-pafs:

Algorithm 7.1. *Let the* diagonal order *for a shape of size $m \times n$ be the following total order on all cell positions:*

$$(1,1),(1,2),(2,1),(1,3),(2,2),(3,1),\ldots$$

That is, order pairs (i,j) with $1 \le i \le m$ and $1 \le j \le n$, first by increasing $i+j$, and then by increasing i.

The smallest-rectangle order *is the total order on the minors of the shape, where minors $\{i_1,i_2\}\times\{j_1,j_2\}$ are first ordered by increasing size (i_2-i_1, j_2-j_1) in the diagonal order, and further ordered by the top-left corner in the diagonal order.*

Assume the filling contains neither $\left(\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$*, nor the* old *pattern (either* $\left(\begin{smallmatrix}\bullet&\circ\\\circ&\circ\end{smallmatrix}\right)$ *or* $\left(\begin{smallmatrix}\circ&\circ\\\circ&\bullet\end{smallmatrix}\right)$*, depending on the direction of the bijection). The bijection will replace instances of the old pattern by instances of the* new *pattern (whichever of* $\left(\begin{smallmatrix}\circ&\circ\\\circ&\bullet\end{smallmatrix}\right)$ *or* $\left(\begin{smallmatrix}\bullet&\circ\\\circ&\circ\end{smallmatrix}\right)$ *is not the old pattern).*

Repeat the following procedure until no instances of the old pattern remain:

(1) *Let M be the first minor with the old pattern, in smallest-rectangle order.*
(2) *Replace M by the new pattern.*

The algorithm terminates because each replacement moves some $\bullet$, depending on the direction of the bijection, strictly down and right, or strictly up and left. Only finitely many such moves are possible. To check that the algorithm is the desired bijection, one needs to verify (i) that a replacement cannot create an instance of $\left(\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$, and (ii) that a replacement cannot create an instance of the old pattern that comes earlier in the smallest-rectangle order. To prove this, attempt to construct a violation, and conclude that it is impossible by connectivity and $\left(\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$-avoidance. We omit the proof details due to space constraints.

It seems plausible, although we have not verified it, that the assumptions needed for this bijection are strictly weaker than the union of top-left and bottom-right CR-erasability, which are both required for the two pp's recurrences to match up.

There is also an independent reason for equivalence between $\left(\begin{smallmatrix}\bullet&\circ\\\circ&\circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\circ&\circ\\\circ&\bullet\end{smallmatrix}\middle|\begin{smallmatrix}\bullet&\circ\\\circ&\bullet\end{smallmatrix}\right)$. The pattern pairs are mapped to each other both by transposition, and by reversing the order of the rows and columns together. Therefore, the pattern pairs will be equivalent in shapes preserved by either transformation (180°-rotation symmetry, and upper-left to bottom-right reflection symmetry).

Empirically, every shape having this equivalence satisfies one of the above conditions: symmetry, or a decomposition into 2×2-components that are connected.

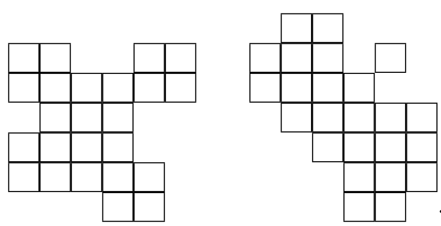
By rotating the patterns and shapes 90°, we see that the analogous bijection works for the pps $\left(\begin{smallmatrix}\circ & \bullet \\ \circ & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\circ & \circ \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\right)$.

8. Empirical Results and Open Problems

8.1. **When are patterns inequivalent?** In the theorems above, we state whether the result appears to be empirically tight. That is, whether we found any shapes that do not satisfy the assumptions of the theorem, but have an equivalence between the corresponding pattern pairs.

Each theorem was tested on a set of about 160 shapes. The data set includes all 2×2-connected 3×3 examples, some Young diagrams, rectangles, skew shapes, other shapes that are horizontally- or vertically- connected, a number of shapes with no apparent regularities, and some shapes that were made up as tests of hypotheses, or counterexamples. Nonetheless, it is a small and unrepresentative set, and claims of empirical tightness should be taken with a grain of salt. We intend to check them more systematically on all shapes up to 5×4, and in particular to see how frequently sporadic coincidences occur. However, we do not have high hopes for results that say when two pps are *not* equivalent.

On the other hand, it is easy to produce shapes so that no two pps are equivalent:



These two examples were found during unsystematic experiments. It may be interesting to try to characterize the set of such shapes. How "regular" can they be? Are they frequent? We conjecture that a large random shape with a sufficiently high density of cells will almost always have this property.

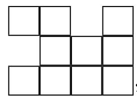
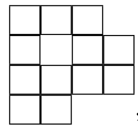
Problem 8.1. Find, as a function of $0 \leq \lambda \leq 1$, the fraction of $n \times n$ shapes with λn^2 cells, which have no equivalent pps.

8.2. **Other equivalences.** Of the $\frac{64 \cdot 63}{2} = 2016$ possible equivalences, we have (partially) described 156 — there are 12 pattern pairs, all connected through $\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)$. Many of the descriptions, especially the indirect ones, are not tight. Here is an example of such a connection:

$$\left(\begin{smallmatrix}\circ & \circ \\ \circ & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ & \circ \\ \bullet & \circ\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}\bullet & \circ \\ \circ & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ & \bullet \\ \circ & \circ\end{smallmatrix}\right). \tag{11}$$

In order for it to work, the shape must be top-right CR-erasable *and* bottom CR-erasable. Requiring both pps to be equivalent to $\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)$ gives excessively strong conditions for the relation between the two. For (11), this is true empirically —

the equivalence holds in many shapes where $\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)$ differs from both:

, , etc.

The bijection in Section 7 is another example. We proved $\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \bullet \\ \bullet & \circ\end{smallmatrix}\right)$ to be equivalent to $\left(\begin{smallmatrix}\bullet & \circ \\ \circ & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)$ for a large class of shapes, in many of which $\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \bullet \\ \bullet & \circ\end{smallmatrix}\right)$ is different from $\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)$. All the skew shapes in our data set fall in this category. Also, as mentioned in Subsection 4.4, the assumptions for the $\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \bullet \\ \bullet & \circ\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)$ equivalence (and its 3 rotations) need to be weakened.

Therefore, there is considerable room for improvement in the description of the 13 pattern pairs in this paper. Moreover, 51 of the pattern pairs are completely untouched. For example, here are some equivalences that appear to hold for all shapes whose 2×2-components are connected (the exact conditions are weaker, and vary by pair):

$$\left(\begin{smallmatrix}\bullet & \circ \\ \circ & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \bullet \\ \circ & \circ\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}\circ & \circ \\ \circ & \bullet\end{smallmatrix}\middle|\begin{smallmatrix}\circ & \bullet \\ \circ & \bullet\end{smallmatrix}\right), \quad \left(\begin{smallmatrix}\bullet & \circ \\ \circ & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \bullet & \circ\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}\circ & \circ \\ \circ & \bullet\end{smallmatrix}\middle|\begin{smallmatrix}\circ & \circ \\ \bullet & \bullet\end{smallmatrix}\right),$$
$$\left(\begin{smallmatrix}\circ & \bullet \\ \circ & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \bullet \\ \circ & \circ\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}\circ & \circ \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \bullet & \circ\end{smallmatrix}\right), \quad \left(\begin{smallmatrix}\circ & \bullet \\ \circ & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ & \bullet \\ \circ & \bullet\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}\circ & \circ \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\circ & \circ \\ \bullet & \bullet\end{smallmatrix}\right).$$

These four symmetrical equivalences look similar to $\left(\begin{smallmatrix}\bullet & \circ \\ \circ & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right) \leftrightarrow \left(\begin{smallmatrix}\circ & \circ \\ \circ & \bullet\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)$ from Section 7, but that bijection does not seem to generalize. Empirically, these 8 pps are not equivalent to $\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\middle|\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)$ except in a handful of shapes with no clear regularity. The recurrence analogous to Lemma 4.3 fails because rows or columns that we would want to delete *can* be involved in forbidden patterns. Thus, a new idea is needed — perhaps a modified recurrence. We believe that any bijection is likely to be quite complex. There are many other coincidences, but the classes of shapes, in which they happen, are either small or unclear, so we will not list them here.

8.3. **Implications between equivalences.** It may not be possible to obtain actual *implications* for the reasons discussed in Subsection 8.1. We can go from shape properties to equivalence, but not from equivalence to shape properties. Nonetheless, the following holds empirically. If any equivalence in the first column of Table 1 on page 8 holds, then so do the other two in its row. The first four equivalences in the second column "imply" the corresponding equivalences in the third column. A little more formally, if we had the weakest possible conditions for all the equivalences, modulo unsystematic coincidences, then conditions in the first column are strictly stronger than those in the second, and those are strictly stronger than the third. Again, this may be extremely difficult to prove because of the "unsystematic coincidences" part.

Something more approachable is to catalog the various unions of the best known conditions for equivalence, and to describe combinatorially each class of shapes in which these unions are true. This would give a poset of shape classes, ordered by inclusion, with a set of equivalences for each.

8.4. **How to count the number of pattern-avoiding fillings?** It may be desirable to know more than just the equivalences between pairs of pps. The recurrences in this paper can be used to get a numeric answer in some circumstances, but there are many more questions left unanswered:

Problem 8.2. Given a set of pattern pairs, is it possible to compute the number of the corresponding pafs in any shape in subexponential, or even polynomial time (in the number of cells in the shape)?

Problem 8.3. Are there explicit formulas counting the number of specific pafs? In specific shapes? (Kitaev et al. answer this for rectangular shapes in [3, 2])

Problem 8.4. Is there a good description of shapes, in which the number of certain pafs is strictly less (greater) than that of another kind of pafs? That is, a theory of pattern-ordering and not just of pattern-equivalence.

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