ON THE n-DIMENSIONAL FRAME

ALEXEY SPIRIDONOV

1. A Historical Introduction

In 1813, Cauchy proved that all convex polyhedra are rigid, and it was widely believed that the same held true for non-convex ones. A self-intersecting flexible construction by Bricard in 1897 raised some doubts about the conjecture. However, it was not until Connelly removed the self-intersections from Bricard's construction in 1977 that the rigidity conjecture was disproved.

It was observed that Connelly's polyhedron, as well as the simpler subsequent construction by Steffen, had constant volume as they flexed. Dennis Sullivan is credited with this discovery. He built a model of a flexible polyhedron, blew smoke into it, saw that none escaped as it flexed, and surmised that the same might be true for all flexible polyhedra. The polyhedron was not behaving as a bellows might, so Connelly dubbed Sullivan's conjecture the "Bellows conjecture". The name stuck.

The conjecture was proved in 1995 by Sabitov in a series of involved, but progressively simpler papers. In 1996 Connelly invited Sabitov to visit Cornell. There, the two, together with Walz (further CSW), produced an elegant proof of the conjecture based on the theory of places.

That laid the original Bellows conjecture to rest. However, Connelly had concurrently formulated a stronger version: all flexed states of a polyhedron are scissor-equivalent. His flexible polyhedron, as well as Steffen's, satisfied this condition. However, both are based on Bricard's octahedron, whose symmetries cause the Dehn invariant to remain constant. Therefore, it's interesting to construct flexible polyhedra unrelated to Bricard's construction.

In 1995, before Sabitov's work became known, V.A. Alexandrov published a paper describing such a construction. His intent was to produce a counterexample to the Bellows conjecture. The polyhedron was based on the "frame", a highly symmetrical zero-volume self-intersecting polyhedron homeomorphic to a torus (described below). He broke its symmetry by introducing variable-sized kinks into the shape; however, this construction fared no better than Bricard's. Both the volume and the Dehn invariant stayed constant during flexing. While the Bellows conjecture is settled, the strong version is not. So, the search for counterexamples, which would likely have to be flexible polyhedra of novel constructions, needs to continue.

Another generalization of the Bellows conjecture is its n-dimensional analog. In 1997, CSW said they thought they knew how to prove it in 4 dimensions, but were stuck for all higher n. Walz gave a talk about the 4-dimensional case at the Canadian Mathematical Society meeting in December 1998, but there appears to be no publications on the matter. The author is not aware of any flexible 4-polytopes that would substantiate or disprove the conjecture. This paper attempts to generalize Alexandrov's frame construction to n dimensions, and thus provide an example of a flexible 4-polytope.

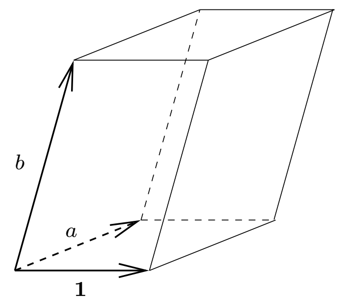

FIGURE 2.1. Flexible chimney: the top and bottom faces are missing.

2. Frame Construction and Properties

Before proceeding to the general construction, we give some low-dimensional examples to familiarize the reader with the language needed for the general case. The content of sections 2.1 and 2.2 is trivial, and might be skipped.

We will abbreviate $0, \ldots 0$, "zeros until the end of the vector", as $\bar{0}$. To check the flexibility of a polytope, we will examine the permissible positions of the vertices of a combinatorial structure under some constraints on the vertex coordinates. To exclude rigid motions, we will fix one of the faces in space. To this end, we fix one of the vertices at zero, one of the incident edges in the line $(x_1, \bar{0})$, a 2d-face incident with this edge in the plane $(x_1, x_2, \bar{0})$, etc. Additionally, without loss of generality, we'll rescale the edge in $(x_1, \bar{0})$ to have length 1.

2.1. **Dimension 2: Parallelogram.**

The nicest 2-dimensional flexible polytope is a parallelogram.

Fix one of the sides as $\mathbf{1} = (1, 0)$ vector from at the origin; then, a parallelogram is completely specified by another vector $a = (a_1, a_2)$, with $a \cdot a = |a|^2 = C \geq 0$. The points on the parallelogram are $0, \mathbf{1}, a, \mathbf{1}+a$ — the sums of all subsets of $\{\mathbf{1}, a\}$.

Fix the C; then, the range of flexion of the parallelogram is given by the solutions of $a \cdot a = C$. These solutions form a one-dimensional family, provided that some solution exists (always), and the Jacobian at this solution has full rank ($\begin{bmatrix} 2a_1 & 2a_2 \end{bmatrix} \neq 0 \Rightarrow a \neq (0, 0)$). Note that when $a = (1, 0)$, we get a line; we'll just consider it (and similar cases in what follows) to be a parallelogram of volume 0.

2.2. **Dimension 3: Chimney.**

We can take a parallelogram and extend it to three dimensions by translating it along a line segment a protruding in the third dimension. A face of the resulting figure will result from translating the side of a parallelogram. Let's say that the fixed face is given by the parallelogram side $\mathbf{1} = (1, 0, 0)$ and the line segment $a = (a_1, a_2, 0)$. Then, the second parallelogram side is some $b = (b_1, b_2, b_3)$. The vertices of the chimney will be all sums of subsets of $\{\mathbf{1}, a, b\}$:

$$0, \mathbf{1}, a, \mathbf{1} + a, b, \mathbf{1} + \mathbf{b}, a + b, \mathbf{1} + a + b,$$

with the edges corresponding to the addition of exactly one new summand (e.g. from a to $a + b$). In this language, a face is specified by a starting point (say $1 + b$), and a pair of vertices such they don't both come from the parallelogram (say $\{a, b\}$). In this example, the vertices of the face are: $1 + b$,$1 + b + a$,$1 + a$,1.

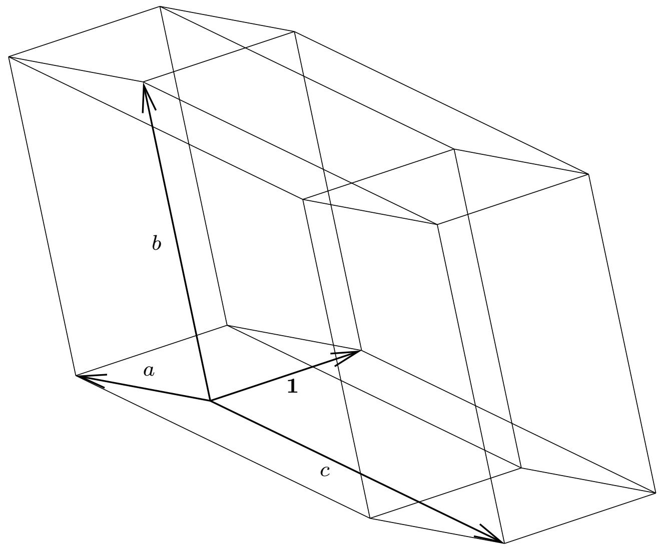

FIGURE 2.2. Alexandrov's frame, enclosing equal amounts of positive and negative volume. The four highlighted vectors parametrize the construction.

This gives us the a chimney shape as in Figure 2.2. Deformations of the chimney are restricted by the following equations:

$$a \cdot a = C, \quad b \cdot b = D \quad \text{(edge lengths stay constant)}$$
$$a \cdot \mathbf{1} = E, \quad a \cdot b = F \quad \text{(face angles stay constant)}$$

So, we have 4 constraints and 5 variables: a_1, a_2, b_1, b_2, b_3. If some solution exists and the Jacobian has full rank for that solution, there is a one-dimensional family of solutions specifying how the chimney flexes. We have a solution as long as the edge lengths are all positive, and the cosine values from the two-edge dot products are in $[-1, 1]$. The Jacobian looks like this:

$$\begin{pmatrix} 2a_1 & 2a_2 & & & \\ & & 2b_1 & 2b_2 & 2b_3 \\ 1 & & & & \\ b_1 & b_2 & a_1 & a_2 & \end{pmatrix} \sim \begin{pmatrix} 1 & & & & \\ & 2a_2 & & & \\ & & a_1 & a_2 & \\ & & & * & 2b_3 \end{pmatrix},$$

where the value of $*$ depends on whether b_1 is zero. It's clear that this has full rank for a generic assignment of the variables.

2.3. **Dimension 3: Frame.** The frame, shown in Figure 2.3, is a torus made from 4 chimneys. Alexandrov's paper breaks the symmetry in this figure in an attempt to create a shape with non-zero, non-constant volume. His symmetry-breaking construction does not even have the limited success of Connelly's crinkle: the volume remains zero. The failure to produce positive volume is not trivial, and results from requiring the desymmetrized shape to be flexible. However, this is beyond the scope of this paper. It should be noted that the frame in Alexandrov's paper is based on a rhomb instead of a parallelogram, although that's not necessary for flexibility.

The frame can be described in the language of the previous sections as follows. We have two pairs of vectors: $\mathbf{1} = (1, 0, 0), a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, 0), c = (c_1, c_2, c_3)$, where 1 and b describe the "fixed" face. As before, we take the vertices to be all sums of subsets of these (for a total of $2^4 = 16$), and the edges to be all pairs of vertices differing in exactly one vector ($\frac{2^4 \cdot 4}{2} = 32$ of them). There are 4 types of faces, corresponding to a shift by a vector from the first pair and a shift by a vector from the second pair; every vertex participates in each available type of face, and each face contains 4 vertices. Hence, the number of faces is $\frac{2^4 \cdot 4}{4} = 16$.

A frame is subject to the following 7 constraints:

$$\begin{array}{ccc} a \cdot a = C, & b \cdot b = D, & \text{(edge lengths stay constant)} \\ & c \cdot c = E & \\ a \cdot b = F, & a \cdot c = G, & \text{(face angles stay constant)} \\ b \cdot \mathbf{1} = H, & c \cdot \mathbf{1} = I & \end{array},$$

and has 8 variables. Again, a solution exists provided the length constants are positive, and the angle constants are valid: then, it's very easy to check that the construction is consistent. The Jacobian (with the $\cdot\mathbf{1}$ rows/columns eliminated) looks like this:

$$\begin{pmatrix} 2a_1 & 2a_2 & 2a_3 & & & & & \\ & & & 2b_1 & 2b_2 & & & \\ & & & & & 2c_1 & 2c_2 & 2c_3 \\ & & & 1 & & & & \\ & & & & & 1 & & \\ b_1 & b_2 & & a_1 & a_2 & & & \\ c_1 & c_2 & & & & a_1 & a_2 & \end{pmatrix}.$$

We apply the permutation 4 5 6 7 8 1 2 3 to the rows & columns of this matrix, and row-reduce to get:

$$\begin{pmatrix} 1 & & & & & & & \\ & & 1 & & & & & \\ a_1 & a_2 & & & & b_1 & b_2 & \\ & & a_1 & a_2 & & c_1 & c_2 & \\ 2b_1 & 2b_2 & & & & & & \\ & & 2c_1 & 2c_2 & 2c_3 & & & \\ & & & & & 2a_1 & 2a_2 & 2a_3 \end{pmatrix} \sim \begin{pmatrix} 1 & & & & & & & \\ & 2b_2 & & & & & & \\ & & 1 & & & & & \\ & & & a_2 & & c_1 & c_2 & \\ & & & & 2c_3 & * & * & \\ & & & & & b_1 & b_2 & \\ & & & & & & * & 2a_3 \end{pmatrix}.$$

Here, $*$ may take different values depending on whether some variables are zero or not. The matrix is full rank for almost all variable assignments; only a set of measure zero is forbidden. Hence, the frame is flexible.

2.4. **The n-Dimensional Frame.**

2.4.1. *Vertices.* Taking cue from the 3-dimensional frame, we construct the n-dimensional frame. The idea is to translate an $(n-1)$-dimensional frame along a parallelogram (that's how we got the 3-dimensional frame from the 2-dimensional one). In this way, we get a frame of higher dimension with a copy of the lower-dimensional frame in every corner of the new frame.

In the language of the previous sections, an n-dimensional frame is defined by $n-1$ pairs of vectors (each additional pair making increasing the number of dimensions):

$$\begin{array}{cc} \mathbf{1} = (1, \bar{0}), & a_1 = (a_{1,1}, \ldots a_{1,n}) \\ a_2 = (a_{2,1}, a_{2,2}, \bar{0}), & a_3 = (a_{3,1}, \ldots, a_{3,n}) \\ a_4 = (a_{4,1}, a_{4,2}, a_{4,3}, \bar{0}) & a_5 = (a_{5,1}, \ldots, a_{5,n}) \\ \cdots & \cdots \\ a_{2n-4} = (a_{2n-4,1}, \ldots, a_{2n-4,n-1}, 0) & a_{2n-3} = (a_{2n-3,1}, \ldots, a_{2n-3,n}) \end{array} \tag{2.1}$$

Here, the vectors $\mathbf{1}, a_2, a_4, \ldots, a_{2n-4}$ define the face of the polytope that we are fixing in space.

Taking all sums of subsets, we get $2^{2(n-1)}$ vertices; generically, these are all distinct. To see this, pick an arbitrarily large K; take some assignments of the vectors, and rescale them all by a factor of $1.00\ldots001$ (with the 1 is in position $K+1$).

Make tiny adjustments to the rescaled vectors so that a_{i1} all have finite decimal expansions, with at most K digits after the decimal point. Starting at decimal position $K+1$, we put a $2(n-1)$-digit marker on all vectors. $\mathbf{1}$ has a 1 in position $K+1$ (because of rescaling) and zeros elsewhere, a_i has a position in $K+1+i$ and zeros elsewhere. Now, for any vertex we can identify the corresponding subset by looking at this marker, so all vertices are distinct. It's clear that a similar procedure can get us a dense set of points such that all vertices are distinct. By continuity (small change in a_i produces a small change in the vertex positions), it follows that the vertices are distinct for almost all variable values.

2.4.2. *Faces.* This polytope has k-dimensional faces specified (not uniquely) by a starting point, and a set of k vectors such that every vector comes from a different pair in (2.1); call these *proper k-sets.* The face is given by vertices corresponding to sums, which fix the remaining $2(n-1)-k$ vectors as in the initial point, and take all 2^k possible subsets for the chosen k vectors.

A proper k-set specifies the shape of a face. The n-dimensional frame has $\binom{n-1}{k}2^k$ k-dimensional face types. Every vertex is in a face of a given type exactly once, each face has 2^k vertices. Letting F be the number of faces, we get the following equality (by counting the vertices with repetition):

$$2^{2(n-1)}\binom{n-1}{k}2^k = 2^k F \Rightarrow F = 2^{2(n-1)}\binom{n-1}{k}$$

2.4.3. *Polytope or Not?* A polytope this is, since the n-dimensional frame is homeomorphic to the n-dimensional torus. The homeomorphism in 2 dimensions is trivial. Assuming we have a homeomorphism for the $n-1$ dimensional frame, we construct the n-dimensional homeomorphism as follows. To construct the n-frame, we're moving the $(n-1)$-frame along a parallelogram. The parallelogram is in some 2-plane, and the $(n-1)$-frame is in a fixed orientation in some hyperplane that intersects, but does not contain the parallelogram plane. As the $(n-1)$-frame moves around the perimeter, it stays in a hyperplane parallel to the original, and retains its original orientation.

We will continuously deform the initial parallelogram and frame to be, respectively, an $(n-1)$-torus and a circle. This corresponds to a continuous deformation of the n-frame. Now, each point on the $(n-1)$-torus T travels along its own circle, all such circles having the same radii, and located in parallel 2-planes. However, the centers of revolution themselves lie on another $(n-1)$-torus, whereas, for a torus we'd like to have them all lie on a line.

To rectify this, we keep the initial $(n-1)$-torus intact, and continuously alter the circles along which its points travel. Pick a hyperline[1] l perpendicular to the plane of our circle of rotation (to avoid self-intersection, put l more than one diameter of T away from T). For every point p on the torus, we continuously move its center of rotation to l, while staying in the plane of p's original circle of rotation. At the end of the transformation, we have a manifold generated by rotating a$(n-1)$-torus T around a hyperline lying in the hyperplane of T. Hence, the n-frame is homeomorphic to an n-torus, and is a compact boundary-less piecewise linear manifold – a polytope.

Incidentally, since our parallelogram is symmetric, and the orientation of the generating $(n-1)$-frame is constant throughout, we end up adding a volume element, and on the reverse side of the parallelogram we add its negative counterpart. Therefore, the volume of the n-frame is zero.

2.4.4. *Flexibility.* The n-dimensional frame has $V_n = \frac{(n-1)n}{2} - 1 + n(n-1) = \frac{3}{2}n(n-1) - 1$ variables, and must obey the following constraints:

$$\begin{array}{cccccc}
\multicolumn{6}{c}{\text{The following edge lengths are constant:}} \\
\multicolumn{6}{c}{a_1 \cdot a_1, \quad a_2 \cdot a_2 \quad \ldots, \quad a_{2n-3} \cdot a_{2n-3}} \\
\multicolumn{6}{c}{\text{The face angles stay constant:}} \\
\mathbf{1} \cdot a_2, & \mathbf{1} \cdot a_3, & \mathbf{1} \cdot a_4, & \ldots, & \mathbf{1} \cdot a_{2n-4} & \mathbf{1} \cdot a_{2n-3} \\
a_1 \cdot a_2, & a_1 \cdot a_3, & a_1 \cdot a_4, & \ldots, & a_1 \cdot a_{2n-4}, & a_1 \cdot a_{2n-3} \\
 & & a_2 \cdot a_4, & \ldots, & a_2 \cdot a_{2n-4}, & a_2 \cdot a_{2n-3} \\
 & & a_3 \cdot a_4, & \ldots, & a_3 \cdot a_{2n-4}, & a_3 \cdot a_{2n-3} \\
 & & \ddots & & & \vdots \\
 & & & & a_{2n-6} \cdot a_{2n-4}, & a_{2n-6} \cdot a_{2n-3} \\
 & & & & a_{2n-5} \cdot a_{2n-4}, & a_{2n-5} \cdot a_{2n-3}
\end{array}$$

That's a total of $C_n = 4 \cdot \frac{(n-2)(n-1)}{2} + 2n - 3 = 2(n-1)(n-1) - 1$ constraints. For $n = 2, 3$, $V_n - C_n = 1$, and so the resulting frame is flexible as argued above.

In dimension 4, $V_n - C_n = 0$, making the system exactly determined. If the above constraints are all independent (for our choice of constants), that implies that it has at most a finite number of solutions. In all higher dimensions, the system is overdetermined with $V_n - C_n < 0$.

A given choice of constants tells us the coordinates for the first face, so $a_2, \ldots a_{2n-4}$ can be considered constant. That leaves us with

$$\frac{3}{2}(n-2)(n-1) + n - 1 = \frac{1}{2}(3n-4)(n-1)$$

[1]An affine subspace of dimension $n-2$, in case I just made this word up.

constraints and $n(n-1)$ variables. Rename the remaining $n-1$ vectors $b_1, \ldots, b_n$. Then, b_i is involved in $n-2$ linear constraints. We'll assume that the constants were chosen so that the fixed face has non-zero volume. In that case, the vectors $\mathbf{1}, a_2, a_4, \ldots, a_{2n-4}$ are linearly independent, and so the $n-2$ linear constraints force each b_i into some fixed plane. Each such plane can be parametrized by some 2 variables $c_i = (c_{i1}, c_{i2})$. On these remaining $2(n-1)$ variables, we have

$$\frac{(n-1)(n-2)}{2} + n - 1 = \frac{n(n-1)}{2}$$

constraints, which are of the form $c_i A_{ij} c_j = C_{ij}$, with A_{ij} some constant $n \times n$ change-of-variable matrix.

So, the flexibility of the frame in dimensions $n \geq 4$ comes down to the independence of the above constraints. There seems to be no reason [2] for this system to have any degrees of freedom, so the n-dimensional frame for $n \geq 4$ should not be flexible. [3] Therefore, to the best of my knowledge, the problem of flexible n-polytopes remains open.

[2]How do I rigorously argue this?

[3]In particular, I do not believe that any special choice of constants will make it flexible.